Practice exam representation theory spring 2018

Question 1. Quaternion group.
Recall the quaterinion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with defining relations $i = jk = -kj$, $j = ki = -ik$, $k = ij = -ji$, $i^2 = j^2 = k^2 = -1$. To avoid confusion we denote the complex number $\sqrt{-1}$ by i.

a. What is the dimension of the group algebra $A = \mathbb{C}[Q_8]$?

b. Write down all the 1-dimensional irreducible representations of A.

c. There is a two-dimensional representation $\rho_2 : A \to End(\mathbb{C}^2)$ given by $\rho_2(-1) = -Id$,

$$\rho_2(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rho_2(j) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

What is $\rho_2(1)$? and $\rho_2(k)$?

d. Show that ρ_2 is indecomposable, is it also irreducible?

e. Explain how the representation $\rho_2^{\otimes}4$ decomposes into irreducible representations.

f. Let H be the subgroup of Q_8 generated by j and W the trivial representation of H. Use the Frobenius reciprocity theorem to compute how many times ρ_2 occurs in the decomposition of the representation $Ind_H^{Q_8} W$.

Question 2. Tensors.
In this exercise U, V, W denote finite dimensional representations of finite group G over an algebraically closed field k of characteristic zero.

a. Prove that $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$ as representations of G.

b. Recall $V \wedge V = V \otimes V / Span(v \otimes v | v \in V)$. Express the character of $V \wedge V$ in terms of the character of V.

c. Is the tensor algebra TV always semi-simple representation of G?

d. Explain why all irreducible finite dimensional representations of $G \times G$ must be isomorphic to tensor products of irreducible representations of G.

Question 3. Symmetric group.

a. How many irreducible representations does the symmetric group S_n have?

b. Given the partition $\lambda = (2, 2)$ of $n = 4$, describe the row and column subgroup of the Young tableau T_λ.

c. Explain how filling the boxes of the Young diagram Y_λ for $\lambda = (2, 2)$ produces a basis for V_λ in a natural way.

d. The symmetric group S_3 acts on the Lie algebra sl_3 of 3×3 matrices with trace 0 by permutation of both rows and columns. Is it true that for any $g \in S_3$ and $a, b \in sl_3$ we have $[g.a, g.b] = [a, b]$?

Question 4. Algebra structure.
Let V be a finite dimensional representation of finite group G.

a. Suppose V is also an algebra. How does the algebra structure on V give rise to an element $a \in V^* \otimes V^* \otimes V$?

b. Does the average $b = \sum_{g \in G} \rho(g)a$ also correspond to an algebra structure on V? Here ρ refers to the G-representation $V^* \otimes V^* \otimes V$.

c. In what sense is b invariant and what does that mean for the corresponding product on V?