Additional exercises representation theory

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Exercise 1
Suppose A is an (associative) algebra with multiplication map $m_A : A \times A \to A$. Let B be the vector space A but the multiplication map $m_B : B \times B \to B$ given by $m_B(x, y) = \frac{m_A(x,y) - m_A(y,x)}{2}$. Does m_B make B into an associative algebra? Is there a unit for B?

Exercise 2
Consider the algebra $A = \mathbb{C}[x, y]/(yx - 1, x^3 - 1)$ and show how it is isomorphic to the group algebra of the cyclic group with three elements. Decompose the regular representation of A into indecomposable representations.

Exercise 3
Does the algebra W over a field k generated by $1, x, y$ subject to the relations $xy - yx = 1$ any finite dimensional representations?

Exercise 4
Describe a two dimensional representation of the group algebra of the alternating group $\mathbb{C}[A_5]$ based on the symmetries of the regular dodecahedron in $\mathbb{R}^3$. Is it irreducible?

Exercise 5
Build an example of a non-trivial finite dimensional algebra (associative with unit) that is NOT isomorphic to the group algebra of some finite group.

Exercise 6
Decompose the regular representation of the group algebra of the Klein four-group over $\mathbb{C}$ into a direct sum of irreducible representations.

Exercise 7
Describe an $(n - 1)$-dimensional irreducible representation of the permutation group S_n by decomposing the representation on k^n that permutes the coordinate vectors. You may assume that the characteristic of the ground field is 0.

Exercise 8
Prove that any finite dimensional representation V of a finite dimensional semisim-

ple algebra A is isomorphic to $\bigoplus_X Hom_A(X, V) \otimes X$. Here X runs over all the irreducible representations of A.

Exercise 9
Do problem 2.15.1 from the book.

Exercise 10
In this exercise we consider Fourier analysis on the cyclic group $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{C}$.

1. Show that all the irreducible characters of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ are given by $k \mapsto e^{\frac{mk}{2\pi n}}$.

2. For any function $f \in F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$ define its Fourier transform $\hat{f} \in F(\mathbb{Z}/n\mathbb{Z}, \mathbb{C})$ by $\hat{f}(m) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{-\frac{mk}{2\pi n}}$. Compute $\hat{1}$ and $\hat{\delta_j}$ where 1 means the constant function and $\delta_j(k) = 1$ if $j = k$ and 0 otherwise.

3. With respect to the inner product $\langle f, g\rangle = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\bar{g}(k)$ on functions, show that Fourier transfrom is a linear isometry.

4. Find a finite analogue of the Poisson summation formula: $\sum_k f(k) = \sum_k \hat{f}(k)$.

Exercise 11
In the case of the partition $\lambda = (2, 2)$ work out explicitly the construction of the element c_λ and the irreducible representation V_λ of S_4. Also work out its character and decompose $V_\lambda \otimes V_\lambda$ into irreducibles.

Exercise 12
Given a finite dimensional algebra A over algebraically closed field k, does the following construction always yield a representation of A? $\rho : A \to k$ given by $\rho(a) = 0$ whenever there exists an n such that $a^n = 0$ and $\rho(a) = 1$ otherwise.