Solutions for Tutorial 5 – Multivariable Analysis

Exercise 2.6.5. Describe all integral curves for the vector field $F(x, y) = (-y, x)$. Answer: all circles centred at the origin.

Exercise 2.6.8. Consider $x' = 1 + x^2$ with $x(0) = 0$ on the interval $(-\pi/2, \pi/2)$. Rewriting this as $x(t) = \int_0^t 1 + x(\tau)^2 d\tau$, we consider $\Phi(\gamma)(t) = \int_0^t 1 + \gamma(\tau)^2 d\tau$. Starting with $\gamma_0 = 0$, we compute

$$\begin{aligned}
\gamma_1(t) = \Phi(\gamma_0)(t) &= \int_0^t 1 d\tau = t, \\
\gamma_2(t) = \Phi(\gamma_1)(t) &= \int_0^t 1 + \tau^2 d\tau = t + t^3/3, \\
\gamma_3(t) = \Phi(\gamma_2)(t) &= \int_0^t 1 + (\tau + \tau^3/3)^2 d\tau = t + t^3/3 + 2\tau^5/15 + t^7/72.
\end{aligned}$$

Exercise 3.3.0.

1. If $\gamma(t) = (1, t, -p, \cos t, \sin t)$ and $\omega = x^1 e^2 + x^4 e^5 \in \Omega^1(\mathbf{R}^5)$, then for $p \in [0, 1]$,

$$\begin{aligned}
\gamma^*\omega(p) &= \omega(\gamma(p)) \circ \gamma'(p) \\
&= \omega(1, p, -p, \cos p, \sin p) \circ (v \longmapsto (0, 1, -1, -\sin(p), \cos(p))v) \\
&= 1 + \cos^2(p).
\end{aligned}$$

   So: $\int_\gamma \omega = \int_{[0,1]} \gamma^*\omega = \int_{[0,1]} (1 + \cos^2(p))dp = (6 + \sin(2))/4$.

2. If $\gamma(s, t) = (2s - t, t + 1)$ and $\omega = e^1 \wedge e^2 \in \Omega^2(\mathbf{R}^2)$, then for $p \in [0, 1]^2$,

$$\begin{aligned}
\gamma^*\omega(p) &= e^1 \circ \gamma'(p) \wedge e^2 \circ \gamma'(p) \\
&= e^1 \circ (v \longmapsto (2v_1 - v_2, v_2)) \wedge e^2 \circ (v \longmapsto (2v_1 - v_2, v_2)) \\
&= (2e^1 - e^2) \wedge e^2 \\
&= 2e^1 \wedge e^2.
\end{aligned}$$

   So: $\int_\gamma \omega = \int_{[0,1]^2} \gamma^*\omega = \int_{[0,1]^2} 2 = 2$.

3. If $\gamma(r, s, t) = (rst, rs, r, 1)$ and $\omega = ze^1 \wedge e^2 \wedge e^3 \in \Omega^3(\mathbf{R}^4)$, then for $p \in [0, 1]^3$ we have $\gamma'(p) = (v \longmapsto (p_2p_3v_1 + p_1p_2v_2 + p_1p_3v_3, p_2v_1 + p_1v_2, v_1, 0))$ and,

$$\begin{aligned}
\gamma^*\omega(p) &= p_1 e^1 \circ \gamma'(p) \wedge e^2 \circ \gamma'(p) \wedge e^3 \circ \gamma'(p) \\
&= p_1(p_2p_3e^1 + p_1p_2e^2 + p_1p_3e^3) \wedge (p_2e^1 + p_1e^2) \wedge e^1 \\
&= -p_1^3 p_3 e^1 \wedge e^2 \wedge e^3.
\end{aligned}$$

   So: $\int_\gamma \omega = \int_{[0,1]^3} \gamma^*\omega = \int_{[0,1]^3} -x^3 z dxdydz = -1/8$.

4. If $\gamma(s,t) = (s/t, 0, 0, 0, 0, 0, 1)$ and $\omega = (x^6 + x^7)e^1 \wedge e^5 \in \Omega^2(\mathbf{R}^7)$, then for $p \in [0,1]^2$ we have $\gamma'(p) = (v \longmapsto (1/tv_1 - s/t^2 v_2, 0, 0, 0, 0, 0, 0))$ and,

$$\gamma^*\omega(p) = (p_6 + p_7)e^1 \circ \gamma'(p) \wedge e^5 \circ \gamma'(p) = (p_6 + p_7)e^1 \circ \gamma'(p) \wedge 0 = 0.$$

So: $\int_\gamma \omega = \int_{[0,1]^2} \gamma^*\omega = \int_{[0,1]^2} 0 = 0$.

Exercise 3.3.3. Let $\varphi : (0,\infty) \times (0, 2\pi) \longrightarrow \mathbf{R}^2$ be $(r,t) \longmapsto (r\cos t, r\sin t)$.

1. For $p \in (0,\infty) \times (0,2\pi)$, we have

$$\begin{aligned}
\varphi^*(dx)(p) = dx \circ \varphi'(p) &= dx \circ (v \longmapsto (\cos(t)v_1 - r\sin(t)v_2, \sin(t)v_1 + r\cos(t)v_2)) \\
&= \cos(t)dr - r\sin(t)dt, \\
d(\varphi^* x)(p) = d(x \circ \varphi)(p) &= d((r,t) \longmapsto r\cos t)(p) \\
&= (v \longmapsto \cos(t)v_1 - r\sin(t)v_2) \\
&= \cos(t)dr - r\sin(t)dt.
\end{aligned}$$

2. If $\gamma(s) = (s,s)$ and $\eta = xdy$, then for $p \in [0,1]$ we have $\gamma^*\eta(p) = \eta \circ \gamma'(p) = pe^2 \circ (v \longmapsto (v,v)) = pdx$. So: $\int_\gamma \eta = \int_{[0,1]} \gamma^*\eta = \int_{[0,1]} xdx = 1/2$.

3. If $\alpha(u) = (u\cos u, u\sin u)$ and $\omega = \frac{-y}{\sqrt{x^2+y^2}}dx + \frac{x}{\sqrt{x^2+y^2}}dy$, then for $p \in [0,1]$ we have $\alpha'(p) = (v \longmapsto (\cos p - p\sin p, \sin p + p\cos p)v)$, and

$$\begin{aligned}
\alpha^*\omega(p) = \omega(\alpha(p)) \circ \alpha'(p) &= \frac{-p\sin p}{|p|}dx + \frac{p\cos p}{|p|}dy \circ \alpha'(p) \\
&= \frac{-p\sin p}{|p|}(\cos p - p\sin p) + \frac{p\cos p}{|p|}(\sin p + p\cos p)dx \\
&= \frac{p}{|p|}dx.
\end{aligned}$$

So: $\int_\alpha \omega = \int_{[0,1]} \alpha^*\omega = 1$.