Solutions for Homework 7 – Multivariable Analysis

Exercise 3.4.1. There are many easy ways to construct 4-cubes γ in $\mathbf{R}^6$ with $\partial\gamma = 0$. Perhaps the easiest is $\gamma(x) = c$ for some $c \in \mathbf{R}^6$ for all $x \in [0,1]^4$. More generally, think of six differentiable $f_i : [0,1]^4 \longrightarrow \mathbf{R}$ with the property that $f_i(0,y,z,w) = f_i(1,y,z,w)$, $f_i(x,0,z,w) = f_i(x,1,z,w)$, and so on. For example

$$\begin{aligned} f_1(x,y,z,w) &= x(x-1)y(y-1), \\ f_2(x,y,z,w) &= z^2(z-1)^3 + 6, \\ f_3(x,y,z,w) &= \sin(2\pi w) \\ &\vdots \end{aligned}$$

Finally, set $\gamma = (f_1, f_2, f_3, f_4, f_5, f_6)$. There are of course even more (complicated) ways to construct γ with $\partial\gamma = 0$.

Exercise 3.5.5. Let $\omega \in \Omega^k(\mathbf{R}^n)$, then $\omega = \sum_{I\in\mathcal{F}} f_I e^I$ for some C^1 functions f_I : $\mathbf{R}^n \longrightarrow \mathbf{R}$ (here, $\mathcal{F} = \{(i_1,\ldots,i_k) : 1 \le i_1 < \cdots < i_k \le n\}$ as denoted in the catch up session). Then we have

$$\begin{aligned} d\omega &= d\sum_{I\in\mathcal{F}} f_I e^I \\ &= \sum_{I\in\mathcal{F}} d(f_I e^I) && \text{by property 3.5.1(1) (linearity of } d\text{)} \\ &= \sum_{I\in\mathcal{F}} df_I \wedge e^I + f_I \wedge d(e^I) && \text{by property 3.5.1(2) (`the product rule')} \\ &= \sum_{I\in\mathcal{F}} df_I \wedge e^I. \end{aligned}$$

This shows that we arrive at the definition of $d\omega$ by only applying Lemma 3.5.1.

Exercise 3.5.6. Consider the 3-cube $F : [0,1]^3 \longrightarrow \mathbf{R}^6$ given by

$$F(x,y,z) = (x, y, z, xyz, x^3 - y, 6x + 3z),$$

and consider the 2-form $\omega \in \Omega^2(\mathbf{R}^6)$ given by

$$\omega = (w^1 + w^2)dw^3 \wedge dw^4 + 2dw^6 \wedge dw^5 + w^2 dw^1 \wedge dw^2.$$

(Here, w^i is the 0-form projecting onto the ith coordinate, with dw^i being its exterior derivative.) By the definition of pull-back as in the lecture notes,

$$\begin{aligned} F^*\omega(w) = {} & ((w^1 + w^2) \circ F)(w)F'(w)^* dw^3(F(w)) \wedge F'(w)^* dw^4(F(w)) \\ & + ((x \longmapsto 2) \circ F)(w)F'(w)^* dw^6(F(w)) \wedge F'(w)^* dw^5(F(w)) \\ & + (w^2 \circ F)(w)F'(w)^* dw^1(F(w)) \wedge F'(w)^* dw^2(F(w)). \end{aligned}$$

Removing the dependence on w,

$$F^*\omega = (x+y)F^*dw^3 \wedge F^*dw^4 + 2F^*dw^6 \wedge F^*dw^5 + yF^*dw^1 \wedge F^*dw^2.$$

Applying property 3.5.1(5),

$$\begin{aligned}
F^*\omega &= (x+y)dF^*w^3 \wedge dF^*w^4 + 2dF^*w^6 \wedge dF^*w^5 + ydF^*w^1 \wedge dF^*w^2 \\
&= (x+y)d(w^3 \circ F) \wedge d(w^4 \circ F) + 2d(w^6 \circ F) \wedge d(w^5 \circ F) + yd(w^1 \circ F) \wedge d(w^2 \circ F) \\
&= (x+y)dz \wedge (xdy + ydz + zdx) + 2(6dx + 3dz) \wedge (3x^2dx - dy) + ydx \wedge dy \\
&= (y-12)dx \wedge dy - (xy + x^2 - 6)dy \wedge dz - (xz + yz + 18x^2)dx \wedge dz.
\end{aligned}$$