Sample Solutions Homework Week 5

Multivariable Analysis

16 December 2019

Problem 2.6.9 (50 pts.)

Describe the integral curves through the point $p = (\pi, \pi, \pi)$ *of the vector field* F *on* $\mathbb{R}^3$ *given by* $F(x, y, z) = e_1 + e_2 + e_3$

Solution: Let P be an open set of $\mathbb{R}^n$ and F a vector field $F : P \to \mathbb{R}^n$. An integral curve γ for F through p is a differentiable map $\gamma : (-\epsilon, \epsilon) :\to P$ for some $\epsilon > 0$ such that $\gamma'(t) = F(\gamma(t))$ for all $t \in (-\epsilon, \epsilon)$ and $\gamma(0) = p$.

Let $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ the map such that $t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. The derivative is given by:

$$\gamma'(t) = (\gamma_1'(t), \gamma_2'(t), \gamma_3'(t)).$$

Then we have $\gamma'(t) = F(\gamma(t)) = e_1 + e_2 + e_3 = (1, 1, 1)$. Now note that $\gamma_1'(t) = 1$ implies that $\gamma_1 = t + c_1$ where c_1 is some constant. Similarly, we find $\gamma_2 = t + c_2$ and $\gamma_3 = t + c_3$.

We can determine the constants by solving $\gamma(0) = (\pi, \pi, \pi)$. This equation yields,

$$\gamma(0) = (c_1, c_2, c_3) = (\pi, \pi, \pi).$$

Therefore, $\gamma(t) = (t + \pi, t + \pi, t + \pi)$.

Problem 3.3.4 (50 pts.)

Compute $\int_\gamma \omega$ *and* $\int_\gamma \eta$ *where* γ *is a 2-cube defined by* $\gamma(s, t) = (s \cos t, \sin t, s)$ and the 2-forms ω, η defined by $\omega = yze^1 \wedge e^3$ and $\eta = yze^3 \wedge e^1$.

Solution: First, we compute:

$$\gamma'(s,t) = \begin{bmatrix} \cos t & -s\sin t \\ 0 & \cos t \\ 1 & 0 \end{bmatrix}.$$

Note that $\gamma_1'(s,t)e_1 = \cos(t)e_1 + e_3$ and $\gamma'(s,t)e_2 = -s\sin(t)e_1 + \cos(t)e_2$. From this we find $\gamma'(s,t)^*e^1 = \cos(t)e^1 - s\sin(t)e^2$, $\gamma'(s,t)^*e^2 = \cos(t)e^2$ and $\gamma'(s,t)^*e^3 = e^1$.

Now we can compute:

$$\begin{aligned}
\gamma^*\omega(s,t) &= \left(\gamma'(s,t)\right)^* \omega\left(\gamma(s,t)\right) \\
&= s\sin(t)\left(\left(\left(\gamma'(s,t)\right)^* e^1\right) \wedge \left(\left(\gamma'(s,t)\right)^* e^3\right)\right) \\
&= s\sin(t)\left(\cos(t)e^1 - s\sin(t)e^2\right) \wedge e^1 \\
&= \left(s^2\sin^2(t)\right) e^1 \wedge e^2.
\end{aligned}$$

By the definition of the integral of a k-form over a 2-cube we find that:

$$\begin{aligned}
\int_\gamma \omega &= \int_{[0,1]^2} \gamma^*\omega(s,t) \\
&= \int_{[0,1]^2} s^2\sin^2(t)dsdt \\
&= \frac{1}{12}(2 - \sin(2)).
\end{aligned}$$

Now we note that by anti-commutativity we get that $\eta = -\omega$. This gives:

$$\begin{aligned}
\int_\gamma \eta &= -\int_\gamma \omega \\
&= -\frac{1}{12}(2 - \sin(2)).
\end{aligned}$$

Therefore, adding the integrals yields:

$$\int_\gamma \eta + \int_\gamma \omega = \frac{1}{12}(2 - \sin(2)) - \frac{1}{12}(2 - \sin(2)) = 0.$$