Solutions for Homework 4 – Multivariable Analysis

Exercise 2.5.7. The assumption tells us that for all $x \neq 0$ we have $|A(x) - x| < \frac{1}{2}|x|$. To show A is invertible, it is enough to show that A is injective, then surjective follows from linear algebra.

Alternative 1. Suppose $x, y \in \mathbf{R}^n$ are distinct. Take a constant $\alpha \in \mathbf{R}$ such that $|\alpha(x-y)| > \frac{1}{2}$. Then, by the reverse triangle inequality,

$$|\alpha(x-y)| - |A(\alpha x) - A(\alpha y)| \leq |A(\alpha(x-y)) - \alpha(x-y)| < \tfrac{1}{2}|\alpha(x-y)|.$$

It follows that

$$|A(\alpha x) - A(\alpha y)| > \tfrac{1}{2}|\alpha(x-y)| > 0.$$

This implies that $A(x) \neq A(y)$, hence A is injective.

Alternative 2. Equivalent to injective is to show that

$$\ker A := \{x \in \mathbf{R}^n : A(x) = 0\} = \{0\}.$$

If $y \neq 0$ but $y \in \ker A$, then $|y| = |A(y) - y| < \frac{1}{2}|y|$. This is false, so $\ker A = \{0\}$.

Exercise 2.5.10. Let $f : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be given by $f(x,y,z) = (x^2 + y^2 + z^2, yz)$. Let us set up the notation to apply the implicit function theorem. Rewrite f as $f : \mathbf{R} \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ with $f(x,y) = (x^2 + (y^1)^2 + (y^2)^2, y^1 y^2)$. Let $x_0 = 0$, let $y_0 = (1,2)$, and define $F : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ by $F(y) = f(x_0, y^1, y^2)$. Computing the matrix of $F'(y_0)$ with respect to the standard basis, we have

$$\begin{pmatrix} \frac{\partial F}{\partial y^1} & \frac{\partial F}{\partial y^2} \end{pmatrix} \Big|_{y=(1,2)} = \begin{pmatrix} 2y^1 & 2y^2 \\ y^2 & y^1 \end{pmatrix} \Big|_{y=(1,2)} = \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}.$$

This has non-zero determinant, so by the implicit function theorem we obtain a function $g : N \longrightarrow M$ for opens $N \subseteq \mathbf{R}$ containing x_0 and $M \subseteq \mathbf{R}^2$ containing y_0 such that

$$(N \times M) \cap f^{-1}\{(5,2)\} = \operatorname{graph} g.$$

Now let us try to solve the equation $f(x,y,z) = (2,1)$ for x near the point $(x_0, y_0, z_0) = (0,1,1)$, since the implicit function theorem fails here (even when permuting the variables). This means we consider the system

$$\begin{aligned} x^2 + y^2 + z^2 &= 2, \\ yz &= 1. \end{aligned}$$

Since $y_0 > 0$ and we are solving the system *locally*, let us only consider $y \in (0, \infty)$, so that we get the equation

$$y^2 + \frac{1}{y^2} = 2 - x^2,$$

or $y^4 + y^2(x^2 - 2) + 1 = 0$. Naively applying the *abc*-formula gives

$$y^2 = \frac{2 - x^2 \pm \sqrt{(x^2-2)^2 - 4}}{2}.$$

We see that this is only valid for $x = 0$, otherwise y^2 is a negative number. It follows that the only local solution of our system around the point $(0, 1, 1)$ *is* in fact the point $(0, 1, 1)$. Hence, we may paramatrise the solution set by a constant map, for example

$$\begin{aligned} g : (-1, 1) &\longrightarrow \mathbf{R}^2, \\ x &\longmapsto (1, 1). \end{aligned}$$