# Sample Solutions Homework Week 3

Multivariable Analysis

2 December 2019

## Problem 2.4.1 (40 pts.)

*Why is there no version of the mean value theorem for $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2$? Give an example of a $C^1$ function $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2$ with $a \neq b$ such that there is no $c$ on the line segment between $a$ and $b$ such that $F(b) - F(a) = F'(c)(b-a)$.*

**Solution:** There exists no such version of the mean value theorem because we cannot guarantee the existence of a unique $c \in (a, b)$ such that:

$$\begin{bmatrix} F_1(b) - F_1(a) \\ F_2(b) - F_2(a) \end{bmatrix} = \begin{bmatrix} F_1'(c)(b-a) \\ F_2'(c)(b-a). \end{bmatrix}.$$

An example where there is no such $c \in (a, b)$ is when we take $a = (0, 0)$ and $b = (2\pi, 2\pi)$ and:

$$F(s, t) = (\cos(t), \sin(t)).$$

Then $F'(s, t) = (-\sin(t), \cos(t))$ and $F(a) - F(b) = (1, 0) - (1, 0) = (0, 0)$. Since $b - a = (2\pi, 2\pi)$, we must have that $F'(c) = 0$. But there does not exist a $c$ in $(a, b)$ such that both $-\sin(t)$ and $\cos(t)$ are zero.

## Problem 2.5.0 (20 pts.)

*Prove that the inverse function theorem in the special case where $x_0 = y_0 = 0$ and $f'(x_0) = \mathrm{Id}_{\mathbb{R}^n}$ implies the inverse function theorem for any $x_0$, $y_0$ and $f'(x_0)$..*

**Solution:** Let us assume that the inverse function theorem holds for $x_0 = y_0 = 0$ and $f'(x_0) = \mathrm{Id}_{\mathbb{R}^n}$. Take a $C^1$ function on an open set $\mathbb{R}^n \supset U \xrightarrow{f} \mathbb{R}^n$, with $f(x_0) = y_0$ and $f'(x_0)$ invertible.

Also take a map $F : U \to \mathbb{R}^n$ such that $F(x) \coloneqq f'(x_0)^{-1}(f(x+x_0) - y_0)$ which is $C^1$. We will check if the inverse function theorem holds for $F$ as well. We note that $F(x_0) = F(0) = f'(x_0)^{-1}(f(x_0) - y_0) = f'(x_0)^{-1}(y_0 - y_0) = 0 = y_0$ and that $U$ is open.

Note that $f'(x_0)^{-1}$ is a linear map, so we get $\left(f'(x_0)^{-1}\right)'(h) = f'(x_0)^{-1}$ for any $h$.

To compute $F'(x)$, we take $h = f(x+x_0) - y_0$ and we use the chain rule to get:

$$F'(x) = \left(f'(x_0)^{-1}\right)'(f(x+x_0) - y_0)f'(x+x_0) = f'(x_0)^{-1}f'(x+x_0).$$

This implies that $F'(x_0) = F'(0) = f'(x_0)^{-1}f'(x_0) = \mathrm{Id}_{\mathbb{R}^n}$. Which has $\det F'(x_0) \neq 0$, so it is invertible. Therefore, the inverse function theorem holds for $F$.

Also note that for $x_0 = y_0 = 0$ and $f'(x_0)^{-1} = \mathrm{Id}_{\mathbb{R}^n}$ we get that $F(x) = f(x)$.

The inverse function theorem tells us there are open sets $0 \in X \subset U$ and $0 \in Y \subset \mathbb{R}^n$ and a $C^1$ function $Y \xrightarrow{G} X$ such that $F \circ G = \mathrm{Id}_Y$ and $G \circ F = \mathrm{Id}_X$.

We also define $g(y) = x_0 + G\left((f'(x_0))^{-1}(f(x+x_0) - y_0)\right)$. Note that $x_0 = y_0 = 0$ and $f'(x_0) = \mathrm{id}_{\mathbb{R}^n}$, which implies:

$$\begin{aligned}
f \circ g &= f\left(x_0 + G\left((f'(x_0))^{-1}(f(x+x_0) - y_0)\right)\right) \\
&= f(x_0 + G(F(x))) \\
&= f(x_0 + G(f(x))) \\
&= f(x_0 + G(y)) \\
&= f(G(y)) \\
&= F(G(y)) \\
&= \mathrm{id}_Y .
\end{aligned}$$

Similarly,

$$\begin{aligned}
g \circ f &= (x_0 + G\left((f'(x_0))^{-1}(f(x+x_0) - y_0)\right)) \quad = G\left((f'(x_0))^{-1}(f(y - y_0)\right) = G(f(y)) \\
&= G(F(y)) \\
&= \mathrm{id}_X .
\end{aligned}$$

Now, we can compute $g'(y)$. To do so use the chain rule to show:

$$f'(g(y))g'(y) = (f \circ g)'(y) = \mathrm{id}_Y'(y) = \mathrm{id}_Y \, .$$

Multiplying both sides with $f'(g(y))^{-1}$, we get:

$$g'(y) = f'(g(y))^{-1}.$$

Which shows that the inverse function theorem holds.

# 1 Problem 3.1.4 (40 pts.)

*Expand* $(ae_1 + ce_2) \wedge (be_1 + de_2)$ *in terms of the basis elements* $e_1 \wedge e_2$*.*

**Solution:**

Note that by multi-linearity of the wedge product we get:

$$(ae_1 + ce_2) \wedge (be_1 + de_2) = (ab)e_1 \wedge e_1 + (ad)e_1 \wedge e_2 + (cb)e_2 \wedge e_1 + (cd)e_2 \wedge e_2.$$

Since $e_1 \wedge e_1 = e_2 \wedge e_2 = 0$ and $e_2 \wedge e_1 = -e_1 \wedge e_2$ we obtain:

$$\begin{aligned}(ae_1 + ce_2) \wedge (be_1 + de_2) &= (ad)e_1 \wedge e_2 - (cb)e_1 \wedge e_2 \\ &= (ad - cb)e_1 \wedge e_2.\end{aligned}$$