Solutions for Homework 2 – Multivariable Analysis

Exercise 2.3.1. In this exercise we identify $L(\mathbf{R}, \mathbf{R})$ with $\mathbf{R}$ by id $\longmapsto 1$, so that derivatives of real functions in a point are simply real numbers. Let $\varphi : [a, b] \longrightarrow \mathbf{R}$ be C^1 with $\varphi'(x) \geq 0$ and $\varphi(b) > \varphi(a)$, and let $f : [\varphi(a), \varphi(b)] \longrightarrow \mathbf{R}$ be continuous. By the fundamental theorem of calculus there exists $F : [\varphi(a), \varphi(b)] \longrightarrow \mathbf{R}$ with $F' = f$. For $x \in [a, b]$, the chain rule then gives

$$(F \circ \varphi)'(x) = f(\varphi(x))\varphi'(x) = ((f \circ \varphi)\varphi')(x).$$

From this equality it becomes clear that $(F \circ \varphi)'$ is continuous, so applying the fundamental theorem of calculus again,

$$\int_{[a,b]} (f \circ \varphi)\varphi' = \int_{[a,b]} (F \circ \varphi)' = F(\varphi(b)) - F(\varphi(a)) = \int_{[\varphi(a),\varphi(b)]} f.$$

Exercise 2.3.2. For $R = [0, 2] \times [0, 3]$ and $f(x, y) = x^2 + y^2$, we compute $\int_R f$ straightforwardly:

$$\begin{aligned}
\int_{[0,2]\times[0,3]} f &= \int_{[0,2]} \int_{[0,3]} f(x, \cdot) && \text{(Fubini)} \\
&= \int_{[0,2]} (x \longmapsto [x^2 y + \frac{y^3}{3}]_0^3) && \text{(FTC)} \\
&= \int_{[0,2]} (x \longmapsto 3x^2 + 9) \\
&= [x^3 + 9x]_0^2 = 26. && \text{(FTC)}
\end{aligned}$$

[1]Note: the notion 'C^1' we have only defined for functions whose domain is an open set, but this can easily be extended. For *any* subset $A \subseteq \mathbf{R}^m$, we say $f : A \longrightarrow \mathbf{R}^n$ is C^1 if there exists a C^1 extension of f, that is, there exists an open $U \subseteq \mathbf{R}^m$ with $A \subseteq U$, and a C^1 function $g : U \longrightarrow \mathbf{R}^n$, such that $g|_A = f$. The same applies when 'C^1' is replaced by 'differentiable'.