Sample Solutions Homework Week 1

Multivariable Analysis

18 november 2019

Problem 2.1.9 (5 pts.)

The set $L(V,W)$ is a vector space when we define addition and scalar multiplication pointwise: $(af+g)(v) = af(v)+g(v)$. Find a basis for $L\left(\mathbb{R}^2,\mathbb{R}^3\right)$ and compute $\dim L\left(\mathbb{R}^2,\mathbb{R}^3\right)$. Same questions for $L(V,W)$ where V,W are arbitrary vector spaces.

Solution: Note that the standard basis for $\mathbb{R}^2$ is $\{e_1,e_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$

and the standard basis for $\mathbb{R}^3$ is given by $\{f_1,f_2,f_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

The elements of $L(\mathbb{R}^2,\mathbb{R}^3)$ are matrices $\left(\phi^i_j\right)$ in $\mathbb{R}^{2\times 3}$, where $\phi e_j = \sum_{i=1}\phi^i_j f_i$.

The standard basis B of $\mathbb{R}^{2\times 3}$ is of the form:

$$B = \left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix}\right\}.$$

The basis B is linearly independent and span the space. The dimension of B is given by $\dim(B) = \dim(\mathbb{R}^{2\times 3}) = 6$.

For the case where V and W are arbitrary vector spaces, we assume that $\{v_1,\ldots,v_n\}$ is a basis for V of dimension n and that $\{w_1,\ldots,w_m\}$ is a basis for W of dimension m.

The space $L(V,W)$ is given by the $m\times n$ matrices. The coefficients of the matrix are given by: ϕ^i_j s.t. $\phi(v_j) = \sum_{i=1}^n \phi^i_j w_i$.

Let E_{ij} be the map that maps $v_j \mapsto w_i$ and $v_k \mapsto 0$ for all $k \neq i$.

Now, we show that $\{E_{ij}\}$ is a basis for $L(V, W)$. Let us take an arbitrary $\phi \in L(V, W)$ and $c \in V$. Then we can write v as a linear combination of its basis vectors i.e. $c = \sum_{i=1}^{n} c^i v_i$. Then:

$$\begin{aligned}
\phi(c) &= \phi\left(\sum_{i=1}^{n} c^i v_i\right) \\
&= \sum_{i=1}^{n} c^i \phi(v_i) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c^i \phi_i^j w_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c^i \phi_i^j E_{ij}(v_j)
\end{aligned}$$

So $\phi = \sum_{j=1}^{m} \phi_j^i E_{ij}$. So $\{E_{ij}\}$ spans $L(V, W)$.

To check linear independence, take $0 = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i^j E_{ij} = 0$. Then apply both sides to v_k for arbitrary k, which leads to:

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i^j E_{ij} = \sum_{j=1} c_k^j w_j$$

Since w_j are the basis vectors for W, we must have that $c_k^j = 0$. Since k was arbitrary, every $c_i^j = 0$. Showing that $\{E_{ij}\}$ are linearly independent and therefore, together with what we have shown before, a basis of $L(V, W)$.

The dimension of $L(V, W) = \dim(L(V, W)) = \dim(\mathbb{R}^{m \times n}) = \dim(W) \dim(V) = mn$.

Problem 2.2.2 (5 pts.)

Compute the derivative $f(1,1)'$ *where* $\mathbb{R}^2 \ni (x,y) \overset{f}{\mapsto} ((x+y)(2x+y), \sin(2x+xy)+y, 1, y-x) \in \mathbb{R}^4$ *using only the properties* $1,2,3,4,5$ *mentioned in Theorem* 2.2.0.1.

Solution:

First define the following maps:

$$\begin{aligned}
f_1 &: (x,y) \mapsto (x+y)(2x+y) \\
f_2 &: (x,y) \mapsto \sin(2x+xy)+y \\
f_3 &: (x,y) \mapsto 1 \\
f_4 &: (x,y) \mapsto y-x.
\end{aligned}$$

By property 4 we have that $f'(1,1)(v,w) = (f_1'(1,1)(v,w), f_2'(1,1)(v,w), f_3'(1,1)(v,w), f_4'(1,1)(v,w))$.

If we compute $(x+y)(2x+y) = 2x^2+3xy+y^2$, so we get:

$$\begin{aligned}
(f_1)'(x,y)(v,w) &= +'(2x^2,3xy,y^2)\left(((x,y)\mapsto 2x^2)'(v,w), ((x,y)\mapsto 3xy)'(v,w), ((x,y)\mapsto y^2)'(v,w)\right) \\
&= ((x,y)\mapsto 2x^2)'(v,w) + ((x,y)\mapsto 3xy)'(v,w) + ((x,y)\mapsto y^2)'(v,w) \\
&= 4xv + 3(xw+yv) + 2yw \\
&= (4x+3y)v + (3x+2y)w
\end{aligned}$$

So, $(f_1)'(1,1)(v,w) = 7v+5w$.

$$\begin{aligned}
f_2'(x,y)(v,w) &= ((x,y)\mapsto \sin(2x+xy)+y)'(v,w) \\
&+'(\sin(2x+xy),y)\left(((x,y)\mapsto \sin(2x+xy))'(v,w), ((x,y)\mapsto y)'(v,w)\right) \\
&= ((x,y)\mapsto \sin(2x+xy))'(v,w) + ((x,y)\mapsto y)'(v,w) \\
&= (\sin(2x+xy))'((x,y)\mapsto 2x+xy)'(v,w) + w \\
&= \cos(2x+xy)(+'(2x+xy)(((x,y)\mapsto 2x)'(v,w), ((x,y)\mapsto xy)'(v,w))) + w \\
&= \cos(2x+xy)(2v+v+w)+w
\end{aligned}$$

So $f_2'(1,1)(v,w) = \cos(3)(3v+w)+w$.

Note that $f_3(x,y)(v,w) = 1$, so $f_3'(1,1)(v,w) = 0$.

Similarly to before, we say:

$$\begin{aligned} f_4'(x,y)(v,w) &= +'(y,-x)\left(((x,y) \mapsto y - x\right)(v,w) \\ &= w - v. \end{aligned}$$

Therefore, combining the above, we get:

$$f'(1,1)(v,w) = \left(7v + 5w, \cos(3)(3v + w) + w, 0, w - v\right).$$