1 Questions Catch-up session

1. Show that $f_1 = e_1 - e_2, f_2 = e_1 + e_2, f_3 = e_3$ is a basis for $\mathbb{R}^3$. Recall that the dual basis f^1, f^2, f^3 satisfies $f^i(f_j) = \delta^i_j$. Express the dual basis $f^1, \dots f^3$ in terms of the basis $e^1, \dots e^3$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y, z) \mapsto (xyz + x^2y, log(xy), \sqrt{z+y}, 4)$. Compute the derivative of f, only using the properties in theorem 2.2.0.1.

3. Show that:

$$\frac{d}{dx} f(g_1(x), \dots, g_k(x)) = \sum_{i=1}^{k} \left(\partial_i f(g_1(x), \dots g_k(x)) \frac{d}{dx} g_i(x) \right).$$

4. Show that $f(x, y) = \left(\sqrt{x^2 + y^2}, \arctan(y/x) \right)$ is invertible for $x \neq 0$.

5. $f : \mathbb{R}^2 \to \mathbb{R}^2 : (u, v) \mapsto (u^3 + uv + v^3, u^2 - v^2)$. Note $f(1, 1) = (3, 0)$, show that f is invertible in a neighbourhood around $(3, 0)$.

6. Prove that there exists an open ball around the point $(1, 1, 1, 1)$ such that the solutions to the system

$$\begin{aligned} w^2 + 2x^2 + y^2 - z^2 &= 3 \\ wxy - xyz &= 0. \end{aligned}$$

   contained in that ball can be written as the graph of a differentiable function g. Give the variables you want g to depend on.

7. Imagine a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(0) = 0$ and consider the level set $S = f^{-1}(\{0\}) = \{x \in \mathbb{R}^n | f(x) = 0\}$. Show that the plane $\ker f'(0)$ is tangent to S in the sense that for any differentiable curve $\gamma : (-a, a) \to S$ we have $\gamma'(0) \in \ker f'(0)$.