Exam multivariable analysis Jan 2020

Exercise 1

a. Prove that the sum of two differential k-forms ω, η is again a differential k-form.

b. If both ω and η are C^1 differentiable, show that their sum is also C^1 differentiable.

c. Under the assumptions of part *b.* prove that $d(\omega + \eta) = d\omega + d\eta$.

Exercise 2

a. For a k-form defined on an open disk centered around the origin, explain why $\partial\gamma = 0$ and $d\omega = 0$ implies $\int_\gamma \omega = 0$. Here γ is any k-chain whose image is inside the above disk.

b. Prove that if f is a real valued continuous function defined on a rectangle $R \subset \mathbb{R}^n$ and $f(r) > 0$ for all $r \in R$ then $\int_R f > 0$. Here $R = \prod_{i=1}^n [a_i, b_i]$ for some $a_i < b_i \in \mathbb{R}$.

c. Suppose we have a continuous function $C \xrightarrow{\Phi} C$ where $C \subset \mathbb{R}^n$ is closed and bounded and there is an $\alpha \in (0,1)$ such that for all $x \neq y \in C$ we have $|\Phi(x) - \Phi(y)| < \alpha|x - y|$. Prove that if p is a fixed point for Φ then for any $x \in C$ the sequence $x, \Phi(x), (\Phi\circ\Phi)(x), (\Phi\circ\Phi\circ\Phi)(x), \ldots$ must converge to p.

Exercise 3

a. Define a 2-form on $\mathbb{R}^3$ by $\omega = xdx \wedge dy + ydx \wedge dz$ and a function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$ given by $f(s,t) = (s, s^2, st)$. Express $f^*\omega$ in terms of ds and dt.

b. Compute $\int_\gamma \omega$ where the 2-cube γ is determined by $\gamma(s,t) = f(s,t)$.

c. Why is there no 1-form α on $\mathbb{R}^3$ such that $d\alpha = \omega$?

Exercise 4
For $0 < m \leq n$ consider the system of m equations in $n + m$ unknowns $x_1, \ldots x_n, y_1, \ldots y_m$:

$$\begin{aligned} x_1 y_1 + y_1 &= 0 \\ x_1 x_2 (y_2)^2 + y_2 &= 0 \\ x_1 x_2 x_3 (y_3)^3 + y_3 &= 0 \\ \ldots\ldots \\ x_1 x_2 x_3 \ldots x_m (y_m)^m + y_m &= 0 \end{aligned}$$

Prove that close to $(x, y) = (0, 0)$, the solution set coincides with the graph of a C^1 function $N \xrightarrow{g} M$. For some open sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$.

Solutions

Exercise 1

a. Prove that the sum of two differential k-forms ω, η is again a differential k-form.
By definition $(\omega+\eta)(p) = \omega(p)+\eta(p)$ and since $\omega(p), \eta(p) \in \Lambda^k(\mathbb{R}^n)^*$ and $\Lambda^k(\mathbb{R}^n)^*$ is a vector space we have $\omega(p) + \eta(p) \in \Lambda^k(\mathbb{R}^n)^*$ showing that the sum is indeed again a k-form.

b. If both ω and η are C^1 differentiable, show that their sum is also C^1 differentiable.
A k-form ω can always be written as $\sum_J \omega_J e^J$ for certain functions $\omega_J : \mathbb{R}^n \to \mathbb{R}$, where the sum runs over all k-element increasing sequences in $\{1, \dots n\}$. The form ω is C^1 if all the f_J are. By definition the sum $\omega + \eta$ can be written as $\omega + \eta = \sum_J(\omega_J + \eta_J)e^J$ so we need to show that the sum of two C^1 functions is again C^1. This follows from the fact that the partial derivative of the sum is the sum of the partial derivatives and also that the sum of two continuous functions is again continuous.

c. Under the assumptions of part b. prove that $d(\omega + \eta) = d\omega + d\eta$.
Using the same notation as above we have by definition $d(\omega + \eta) = d\sum_J(\omega_J + \eta_J)e^J = \sum_J d(\omega_J + \eta_J) \wedge e^J = \sum_J(d\omega_J + d\eta_J)) \wedge e^J = \sum_J d\omega_J \wedge e^J + \sum_J d\eta_J \wedge e^J = d\omega + d\eta$.

Exercise 2

a. For a k-form defined on an open disk centered around the origin, explain why $\partial\gamma = 0$ and $d\omega = 0$ implies $\int_\gamma \omega = 0$. Here γ is any k-chain whose image is inside the above disk.
Since $d\omega = 0$ and we are on the disk the Poincare lemma applies and there exists a $(k-1)$-form α such that $\omega = d\alpha$. Next Stokes theorem says that $\int_\gamma \omega = \int_\gamma d\alpha = \int_{\partial\gamma} \alpha = 0$ since $\partial\gamma = 0$.

b. Prove that if f is a real valued continuous function defined on a rectangle $R \subset \mathbb{R}^n$ and $f(r) > 0$ for all $r \in R$ then $\int_R f > 0$. Here $R = \prod_{i=1}^n [a_i, b_i]$ for some $a_i < b_i \in \mathbb{R}$.
One of the properties of the integral is that it is bounded from below by the minimum of the function times the volume of the rectangle. Since the rectangle is closed and bounded the minimum is actually attained and it is positive since f is everywhere positive. The volume of the rectangle is non-zero by assumption.

c. Suppose we have a continuous function $C \xrightarrow{\Phi} C$ where $C \subset \mathbb{R}^n$ is closed and bounded and there is an $\alpha \in (0, 1)$ such that for all $x \neq y \in C$ we have $|\Phi(x) - \Phi(y)| < \alpha|x - y|$. Prove that if p is a fixed point for Φ then for any $x \in C$ the sequence $x, \Phi(x), (\Phi\circ\Phi)(x), (\Phi\circ\Phi\circ\Phi)(x), \dots$ must converge to p.
Since $\Phi(p) = p$ and if $\Phi^n(x)$ means applying Φ to x n times we have $|\Phi^n(x) - p| < \alpha^n|x - p|$. Choose $\epsilon > 0$ then $|\Phi^n(x) - p| < \epsilon$ when $n > \log_\alpha \frac{\epsilon}{|x-p|}$.

Exercise 3

a. Define a 2-form on $\mathbb{R}^3$ by $\omega = xdx \wedge dy + ydx \wedge dz$ and a function $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$ given by $f(s, t) = (s, s^2, st)$. Express $f^*\omega$ in terms of ds and dt.
Since $f^*d = df^*$ we have $dx = ds$ and $dy = 2sds$ while $dz = sdt + tds$ so $f^*\omega = sds \wedge 2sds + s^2ds \wedge (sdt + tds) = s^3ds \wedge dt$.

b. Compute $\int_\gamma \omega$ where the 2-cube γ is determined by $\gamma(s,t) = f(s,t)$.

$$\int_\gamma \omega = \int_{I^2} s^3 ds \wedge dt = \int_{[0,1]^2} s^3 = \int_{[0,1]} s^3 = \frac{1}{4}$$

c. Why is there no 1-form α on $\mathbb{R}^3$ such that $d\alpha = \omega$?
Because $d\omega = dy \wedge dx \wedge dz \neq 0$.

Exercise 4
For $0 < m \leq n$ consider the system of m equations in $n+m$ unknowns $x_1, \dots x_n, y_1, \dots y_m$:

$$\begin{aligned} x_1 y_1 + y_1 &= 0 \\ x_1 x_2 (y_2)^2 + y_2 &= 0 \\ x_1 x_2 x_3 (y_3)^3 + y_3 &= 0 \\ \dots\dots \\ x_1 x_2 x_3 \dots x_m (y_m)^m + y_m &= 0 \end{aligned}$$

Prove that close to $(x,y) = (0,0)$, the solution set coincides with the graph of a C^1 function $N \xrightarrow{g} M$. For some open sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$.

Writing the solution to the system of equations as $f^{-1}(0)$ with f given by

$$f(x,y) = \sum_{i=1}^{m} ((\prod_{j=1}^{i} x_j)(y_i)^i + y_i)e_i$$

we see that with $F(y) = f(0,y)$ we get $F(0) = Id$ and so $F'(0)$ is also the identity which is invertible. By the implicit function theorem this means that the solution set $f^{-1}(0)$ is locally equal to the graph of a C^1-function $N \xrightarrow{g} M$. For some open sets $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$.