SOLUTIONS TO THE PRACTICE EXAM

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Question 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + x$.

a) The Jacobian $(Jf)(p)$ of f at $p \in \mathbb{R}$ is:

$$(Jf)(p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i.j}|_p = (3x^2+1)|_p = (3p^2+1)$$

(i.e. $(Jf)(p)$ is a 1 by 1 matrix.) Now, if we take Jf as the linear approximation, we have for the error term, for all $p, h \in \mathbb{R}$:

$$\epsilon = f(p+h) - f(p) - (Jf)(p)h = (p+h)^3 + p + h - p^3 - p - (3p^2+1)h = 3ph^2 + h^3 = o(h)$$

Hence f is differentiable and we have $f'(p) = (Jf)(p) \in L(\mathbb{R}, \mathbb{R})$.

b) Note that the partial derivative given by $\frac{\partial f}{\partial x}(p) = 3p^2 + 1$ is continuous in p, hence f is C^1 (even C^∞). Moreover $\det(f'(p)) = 3p^2 + 1 \geq 1$ for all p. Then from the Inverse Function Theorem it follows that f locally is a diffeomorphism. Since $\mathbb{R}$ is connected, this means that f globally is a diffeomorphism. (The global inverse can be reconstructed from glueing together the local pieces given by the theorem.)

Question 2.

$$S^{1+} = \{(x,y) \in S^1 | x, y \geq 0\}$$
$$S^{1+\prime} = \{(x,y) \in S^1 | x, y > 0\}$$

a) $S^1 = f^{-1}(1)$ is the level set of the C^1-function $\mathbb{R}^2 \ni (x,y) \overset{f}{\mapsto} x^2 + y^2 \in \mathbb{R}$. Moreover $\ker'(p)$ is one-dimensional for any point $p \in S^1$. Therefore we may apply the implicit function theorem at every point to construct an atlas of S^1 and such atlasses always give rise to manifolds as shown in the lecture notes.

$S^{1+\prime}$ is an open subset of S^1. Any open subset U of any n-manifold M is again a manifold. This is because the Hausdorff, second countable and locally homeomorphic to R^n properties are clearly inherited by the open subset. An atlas with charts U^α is obtained from the atlas of M by setting $U^\alpha = M^\alpha \cap q^{-1}(q(M^\alpha) \cap U)$. Since the quotient map q restricted to M^α is a homeomorphism we see that U^α is an open subset of R^n. The transition maps of M are restricted to U^α to finish the atlas of U.

Alternatively, a more explicit approach to solve the same exercise is as follows: We can embed $S^{1+\prime}$ in $\mathbb{R}$ via the homeomorphism $\phi\colon S^{1+\prime} \to (0,1)$ given by $\phi(x,y) = x$. To see that ϕ is a homeomorphism consider the following. ϕ is a bijection since its inverse is given by $\phi^{-1}(x) = (x, \sqrt{1-x^2})$. It is clear that ϕ^{-1} is continuous as combination of continuous maps. Now let $U \subset (0,1)$ be open. Define $X = \{(x,y) \in \mathbb{R}^2 | x \in U\}$. Then X is clearly open in $\mathbb{R}^2$. We have $\phi^{-1}(U) = X \cap S^{1+\prime}$. Since $S^{1+\prime}$ has the subspace topology inherited from $\mathbb{R}^2$, we see that $\phi^{-1}(U)$ is open and thus that ϕ is continuous. Hence ϕ is a homeomorphism.

Since subspaces of $\mathbb{R}$ are Hausdorff and second countable, this means that $S^{1+\prime}$ is a C^1 one-dimensional manifold given by an atlas consisting of only the one chart given above.

We cover S^1 by the open subsets $W^1 = S^1\backslash\{(1,0)\}$ and $W^2 = S^1\backslash\{(-1,0)\}$. W^1 and W^2 are homeomorphic to $\mathbb{R}$. To see this we use stereographic projection on the line $L = \{x = 0\} \subset \mathbb{R}^2$. We construct the map $\psi\colon W^1 \to \mathbb{R}$ in the following way. For $p \in W^1$ consider the line $K_p \subset \mathbb{R}^2$ that goes through p and $(1,0)$. Then L and K_p intersect at a point and we take $\psi(p)$ to be the y-coordinate of that point. If we write $p = (p_x, p_y)$ then K is given by the equation $y = \frac{p_y}{p_x-1}x - \frac{p_y}{p_x-1}$. Hence we have $\psi(p_x,p_y) = -\frac{p_y}{p_x-1}$. Similarly we construct $\chi\colon W^2 \to \mathbb{R}$ and have $\chi(p_x,p_y) = \frac{p_y}{p_x+1}$. In a similar way we find that their inverses are given by $\psi^{-1}(t) = (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ and $\chi^{-1}(t) = (\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1})$. It is clear that ψ^{-1} and χ^{-1} are continuous as combination of continuous maps. The fact that ψ and χ are continuous follows from a similar construction as for ϕ. Namely for a $U \subset \mathbb{R}$ open, we can contruct an open V subset of $\mathbb{R}^2$ whose intersection with W^1 or W^2 equals the preimage of U. Just identify L with $\mathbb{R}$ and take for V the union of all the lines through a point in U and $(1,0)$ in the case of ψ and $(-1,0)$ in the case of χ, minus the nasty points $(1,0)$ or $(-1,0)$. Hence ψ and χ are continuous and thus homeomorphisms.

Now we have $\psi((-1,0)) = 0$ and $\chi((1,0)) = 0$. Hence we define the transition maps $\tau_2^1, \tau_1^2\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ by $\tau_2^1 = \chi|_{W^1\cap W^2} \circ \psi^{-1}|_{\mathbb{R}\backslash\{0\}}$ and $\tau_1^2 = \psi|_{W^1\cap W^2} \circ \chi^{-1}|_{\mathbb{R}\backslash\{0\}}$. It is clear that these maps satisfy the cocycle condition in Definition 30. Furthermore we have:

$$\tau_2^1(t) = (\chi|_{W^1\cap W^2} \circ \psi^{-1}|_{\mathbb{R}\backslash\{0\}})(t) = \chi|_{W^1\cap W^2}(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}) = \frac{2t}{t^2+1} \cdot (\frac{t^2-1}{t^2+1} + 1)^{-1} = \frac{2t}{2t^2} = \frac{1}{t}$$

$$\tau_1^2(t) = (\psi|_{W^1\cap W^2} \circ \chi^{-1}|_{\mathbb{R}\backslash\{0\}})(t) = \psi|_{W^1\cap W^2}(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}) = -\frac{2t}{t^2+1} \cdot (\frac{t^2-1}{t^2+1} - 1)^{-1} = \frac{-2t}{-2} = t$$

which are both C^1 (even C^∞). Since they are each others inverses, they are diffeomorphisms. By construction of the transition maps we have $((\mathbb{R}\backslash\{0\}) \sqcup (\mathbb{R}\backslash\{0\}))/\sim = S^1$.

S^1 is thus a one-dimensional manifold that can be realised by the atlas given by two charts of the form $\mathbb{R}\backslash\{0\}$ and the transition maps as given above.

b) Note that $S^{1+'} \times W^1$ and $S^{1+'} \times W^2$ together cover $S^{1+'} \times S^1$. Consider therefore the charts $M^1 = M^2 = (0,1) \times \mathbb{R}$ with overlaps $M_2^1 = M_1^2 = (0,1) \times (\mathbb{R}\backslash\{0\})$ and transition functions given by $\rho_2^1 = id \times \tau_2^1$ and $\rho_1^2 = id \times \tau_1^2$. Since the τ and id are C^1 diffeomorphisms, the ρ are by construction also C^1 diffeomorphisms. Since $id_{S^{1+'}} = \phi \circ \phi^{-1}$, we see that the ρ arise as transition maps from the cover $\{S^{1+'} \times W^1, S^{1+'} \times W^2\}$. Hence they satisfy the cocycle conditions of Definition 30 and by conctruction we have $(\Big((0,1) \times (\mathbb{R}\backslash\{0\})\Big) \sqcup \Big((0,1) \times (\mathbb{R}\backslash\{0\})\Big))/\sim = S^{1+'} \times S^1$.

Hence $S^{1+'} \times S^1$ is a two-dimensional C^1 manifold and can be realised by the atlas given by two charts of the form $(0,1) \times (\mathbb{R}\backslash\{0\})$ and the transition maps ρ as given above.

c) Let (e^1, e^2, e^3, e^4) denote the standard basis of $(\mathbb{R}^4)^*$. Then it is clear that the vectors $e^1 \wedge e^2$, $e^1 \wedge e^3$, $e^1 \wedge e^4$, $e^2 \wedge e^3$, $e^2 \wedge e^4$ and $e^3 \wedge e^4$ are linearly independent, since $\wedge$ is an alternating bilear product. Since $\dim(\Lambda^2(\mathbb{R}^4)^*) = \binom{4}{2} = 6$, these six vectors span $\Lambda^2(\mathbb{R}^4)^*$ and thus form a basis for $\Lambda^2(\mathbb{R}^4)^*$.

d) We have:

$$d(x^1(1+x^4)dx^3) = (1+x^4)dx^1 \wedge dx^3 - x^1 d(1+x^4)dx^3 = (1+x^4)dx^1 \wedge dx^3 - x^1 dx^4 dx^3$$

We can parametrize $S^{1+}\times S^1$ by $\gamma\colon [0,1]^2 \to S^{1+}\times S^1$ given by $\gamma(s,t) = (s, \sqrt{1-s^2}, \cos 2\pi t, \sin 2\pi t)$. On the boundary we have:

$$\begin{aligned}
\gamma_{1,0}(u) &= (0, 1, \cos 2\pi u, \sin 2\pi u)\\
\gamma_{1,1}(u) &= (1, 0, \cos 2\pi u, \sin 2\pi u)\\
\gamma_{2,0}(u) &= (u, \sqrt{1-u^2}, 1, 0)\\
\gamma_{2,1}(u) &= (u, \sqrt{1-u^2}, 1, 0)
\end{aligned}$$

We have:

$$\partial\gamma = \sum_{i=1}^{2}\sum_{\sigma\in\{0,1\}}(-1)^{i+\sigma}\gamma_{i,\sigma} = \gamma_{1,1} - \gamma_{1,0}$$

Stokes Theorem thus gives:

$$\begin{aligned}
\int_{S^{1+}\times S^1}(1+x^4)dx^1\wedge dx^3 - x^1dx^4dx^3 &= \int_\gamma d(x^1(1+x^4)dx^3) = \int_{\partial\gamma} x^1(1+x^4)dx^3\\
&= \int_{\gamma_{1,1}} x^1(1+x^4)dx^3 - \int_{\gamma_{1,0}} x^1(1+x^4)dx^3\\
&= \int_0^1 x^1(\gamma_{1,1}(u))(1+x^4((\gamma_{1,1}(u))))I(\gamma_{1,1}'(u), dx^3(u))du\\
&\quad - \int_0^1 x^1(\gamma_{1,0}(u))(1+x^4((\gamma_{1,0}(u))))I(\gamma_{1,0}'(u), dx^3(u))du\\
&= \int_0^1 x^1(\gamma_{1,1}(u))(1+x^4((\gamma_{1,1}(u))))I(\gamma_{1,1}'(u), dx^3(u))du\\
&= \int_0^1 (1+\sin 2\pi u)I(\gamma_{1,1}'(u), dx^3(u))du\\
&= \int_0^1 (1+\sin 2\pi u)I(\begin{pmatrix}0\\0\\-2\pi\sin 2\pi u\\2\pi\cos 2\pi u\end{pmatrix}, \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix})du\\
&= \int_0^1 (1+\sin 2\pi u)\det\begin{pmatrix}-2\pi\sin 2\pi u\end{pmatrix}du\\
&= -2\pi\int_0^1 \sin 2\pi u(1+\sin 2\pi u)du\\
&= -2\pi\int_0^1 \sin^2 2\pi u du = -\int_0^{2\pi}\sin^2 u du = -\pi
\end{aligned}$$

Actually, since no orientation was given, this answer is determined up to sign. If a different parametrization of $S^{1+}\times S^1$ was chosen, the sign of the integral could have changed.

Question 3. Let $\mathbb{H} \subset \mathbb{C}$ be the hyperbolic plane and consider the map $\phi\colon \mathbb{H} \to \mathbb{H}$ given by $\phi(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$. Write $z = u + iv$ with $u, v \in \mathbb{R}$, then $v > 0$. Note that c and d cannot be both zero and that $Im(z) > 0$, hence $cz + d \neq 0$ for all c, d and z. Moreover we have:

$$\begin{aligned}
Im(\phi(z)) &= Im(\frac{az+b}{cz+d}) = Im(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}) \\
&= Im(\frac{(au+b+iav)(cu+d-icv)}{|cz+d|^2}) \\
&= \frac{acuv+adv-acuv-bcv}{|cz+d|^2} = \frac{v}{|cz+d|^2} > 0
\end{aligned}$$

hence ϕ is well defined. ϕ is continuous, since it's a combination of continuous maps.

Its inverse is given by $\phi^{-1}(z) = \frac{b-dz}{cz-a}$. Note that a and c cannot be both zero and that $Im(z) > 0$, hence $cz - a \neq 0$ for all a, c and z. Moreover we have:

$$\begin{aligned}
Im(\phi^{-1}(z)) &= Im(\frac{b-dz}{cz-a}) = Im(\frac{(b-dz)(c\overline{z}-a)}{|cz-a|^2}) \\
&= Im(\frac{(b-du-idv)(cu-a-icv)}{|cz-a|^2}) \\
&= \frac{-cduv+adv-bcv+cduv}{|cz-a|^2} = \frac{v}{|cz-a|^2} > 0
\end{aligned}$$

hence ϕ^{-1} is well defined. Thus we conclude that ϕ is bijective. ϕ^{-1} is also continuous, since it's a combination of continuous maps.

We have as their complex derivatives $\phi', (\phi^{-1})'\colon \mathbb{H} \to L(\mathbb{C}, \mathbb{C})$:

$$\begin{aligned}
\phi'(z) &= \frac{a}{cz+d} - \frac{c(az+b)}{(cz+d)^2} = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2} \\
(\phi^{-1})'(z) &= \frac{-d}{cz-a} - \frac{c(b-dz)}{(cz-a)^2} = \frac{-d(cz-a)-c(b-dz)}{(cz-a)^2} = \frac{1}{(cz-a)^2}
\end{aligned}$$

which are both well defined maps. (The linear maps act on $\mathbb{C}$ by multiplication just as in 1a).) The error terms of these derivatives give are:

$$\begin{aligned}
\phi(z+h) - \phi(z) - \phi'(z)h &= \frac{a(z+h)+b}{c(z+h)+d} - \frac{az+b}{cz+d} - \frac{h}{(cz+d)^2} \\
&= \frac{(a(z+h)+b)(cz+d)^2 - (az+b)(cz+d)(c(z+h)+d) - h(c(z+h)+d)}{(c(z+h)+d)(cz+d)^2} \\
&= \frac{ah(cz+d)^2 - ch(az+b)(cz+d) - h(c(z+h)+d)}{(c(z+h)+d)(cz+d)^2} \\
&= h \cdot \frac{(a(cz+d)-c(az+b)-1)(cz+d) - ch}{(c(z+h)+d)(cz+d)^2} \\
&= -h^2 \cdot \frac{c}{(c(z+h)+d)(cz+d)^2} = o(h)
\end{aligned}$$

$$\begin{aligned}
\phi^{-1}(z+h)-\phi^{-1}(z)-(\phi^{-1})'(z)h &= \frac{b-d(z+h)}{c(z+h)-a}-\frac{b-dz}{cz-a}-\frac{h}{(cz-a)^2}\\
&= \frac{(b-d(z+h))(cz-a)^2-(b-dz)(cz-a)(c(z+h)-a)-h(c(z+h)-a)}{(c(z+h)-a)(cz-a)^2}\\
&= \frac{-dh(cz-a)^2-ch(b-dz)(cz-a)-h(c(z+h)-a)}{(c(z+h)-a)(cz-a)^2}\\
&= h\cdot\frac{(-d(cz-a)-c(b-dz)-1)(cz-a)-ch}{(c(z+h)-a)(cz-a)^2}\\
&= -h^2\cdot\frac{c}{(c(z+h)-a)(cz-a)^2}=o(h)
\end{aligned}$$

hence the given derivatives are indeed correct. Moreover they are both continuous, since they're both a combination of continuous maps. Hence ϕ and ϕ^{-1} are C^1, so ϕ is a diffeomorphism.

Let $z=u+iv\in\mathbb{H}$ and $p,q\in\mathbb{C}$. The pullback of the hyperbolic metric by ϕ is:

$$\begin{aligned}
(\phi^*g_{hyp})(z)(p,q)=g_{hyp}(\phi(z))(\phi'(z)p,\phi'(z)q) &= \frac{1}{Im(\phi(z))^2}\cdot Re(\phi'(z)p\overline{\phi'(z)q})\\
&= \frac{1}{Im(\phi(z))^2}\cdot Re(|\phi'(z)|^2p\overline{q})=\frac{|\phi'(z)|^2}{Im(\phi(z))^2}\cdot Re(p\overline{q})\\
&= \frac{1}{|cz+d|^4}\cdot(\frac{v}{|cz+d|^2})^{-2}\cdot Re(p\overline{q})=v^{-2}\cdot Re(p\overline{q})\\
&= \frac{1}{Im(z)^2}\cdot Re(p\overline{q})=g_{hyp}(z)(p,q)
\end{aligned}$$

Hence ϕ is an isometry.

Question 4. Let M be an m-dimensional smooth (i.e. C^∞) manifold and let E be a vector bundle over M. Let $s\colon M\to E$ be a section of E. Denote by $\pi\colon E\to M$ the projection map (as defined in Lemma 26). We have $\pi\circ s=id_M$. Taking derivatives gives maps $s'\colon TM\to TE$, $\pi'\colon TE\to TM$ and $id'_M\colon TM\to TM$. Note that id_M is locally a linear map and thus equals its own derivative, i.e. we have locally $(id'_M)^{\alpha,\beta}(p,v)=(id_M^{\alpha,\beta}(p),(id_M^{\alpha,\beta})'(p)v)=(id_M^{\alpha,\beta}(p),id_{\mathbb{R}^m}v)=(p,v)=id_{TM}^{\alpha,\beta}(p,v)$. Hence we have $id'_M=id_{TM}$.

From the local description of a derivative as a map on the tangent bundles and using the defined given in the question we can write the following. Let $f\colon M\to N$ be a map between manifolds. Then $f'\colon TM\to TN$ is given by $f'(p,v)=(f(p),f'(p)(v))$, where $f'(p)$ is a linear map.

Let $(p,v)\in TM$. Applying the chain rule gives:

$$\begin{aligned}
(p,v)=id_{TM}(p,v)=id'_M(p,v) &= (\pi\circ s)'(p,v)\\
&= (\pi'\circ s')(p,v)=\pi'(s(p),s'(p)(v))\\
&= ((\pi\circ s)(p),\pi'\big(s(p)\big)\big(s'(p)(v)\big))\\
&= (p,(\pi'(s(p))\circ s'(p))(v))
\end{aligned}$$

Hence we find $I=\pi'(s(p))\circ s'(p)$, where I is the identity linear map, and we have $\ker(s'(p))\subset\ker(\pi'(s(p))\circ s'(p))=\ker I=\{0\}$ for all $p\in M$. We conclude that $\ker(s'(p))=\{0\}$ and thus that $s'(p)$ is injective for all $p\in M$. s is thus an immersion.

Note on the geometrical intuition: Geometrical intuition often guides the ideas of proofs. My intuition behind the previous proof is the following. The tangent bundle can be thought of as attaching a hyperplane at every point of the manifold such that the hyperplane is tangent to the manifold at that point. To see this note that locally the manifold M looks like $\mathbb{R}^m$ and the tangent bundle TM locally is given by $M^\alpha \times \mathbb{R}^m$, i.e. locally the manifold and the attached hyperplane $\mathbb{R}^m$ 'coincide' and thus are tangent to each other. Now the derivative of a map acts as a linear map on the fibres (an other word for the attached hyperplanes (the V's in Definition 37)). An immersion is a map for which the action of the derivative is injective on the fibres.

We want to prove that a section is an immersion. Suppose this is not the case. Then there is a $p \in M$ such that the linear map $s'(p)$ is not injective and thus has a non trivial kernel. This kernel is some linear subspace of the fibre $\mathbb{R}^m$ (at p). Since we can think of a fibre of the tangent bundle as being tangent to the manifold, the kernel of $s'(p)$ thus points us to directions on the manifold along which s is constant around p (locally). (A function having a zero derivative at a point can be thought of as the function being locally constant around that point by considering linear approximations.) This would mean that there are points near p that also get mapped to $s(p)$ by s. This however violates the idea that s can be locally given as the graph of a map $s^\alpha \colon M^\alpha \to V$, where $M^\alpha \times V = E^\alpha$, since there thus are points $q \neq p$ near p that get mapped to $s(p) = (p, s^\alpha(p))$ in the local graph of s, i.e. s does not send these points to a point 'above' themselves in the graph as it should ($s(q) = (p, \cdot)$ in stead of $s(q) = (q, \cdot)$).

This thus causes trouble in the fact that we should have $\pi \circ s = id_M$. This indicates that this is the expression you would like to examine. Our previous argument however hinged on the fact that we considered linear approximations and therefore could take s to be locally constant. This suggests that linearising $\pi \circ s = id_M$ by taking a derivative and looking at the corresponding linear maps is a good idea.

Note that a geometric intuition does not constitute a proof, but it can help you find one when you are stuck on a question.