SOLUTION TO WORK GROUP ASSIGNMENT 3.12

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Before showing the solution to the problem I want to shed some light on free vector spaces and why they are necessary for the construction of the vector spaces we consider. First the definition of a free vector space. If S is a set and $\mathbb{F}$ a field (a field is a set of numbers including 0 and 1 on which addition, subtraction, multiplication and division are defined) then the free vector space $Span_{\mathbb{F}}(S)$ of S over $\mathbb{F}$ is the set:

$$Span_{\mathbb{F}}(S) = \{f\colon S \to \mathbb{F} | f^{-1}(\mathbb{F}\backslash\{0\}) \text{ is finite}\}$$

It is clear that this set inherits an $\mathbb{F}$-vector space structure from $\mathbb{F}$ by performing operations pointwise. Furthermore an element $f \in Span_{\mathbb{F}}(S)$ is determined by its values on S. We can therefore represent it as:

$$f = \sum_{s \in f^{-1}(\mathbb{F}\backslash\{0\})} f(s)s$$

This means that elements of $Span_{\mathbb{F}}(S)$ can be viewed as formal finite linear combinations in the elements of S with coefficients in $\mathbb{F}$.

Now as to why these vector spaces are needed for our constructions. What we want to do is redefine the linear structure on the vector spaces $V \times V$ and $L(\mathbb{R}^2, V)$. This is normally done by quotienting out some linear equivalence relation. However if we do this while $V \times V$ and $L(\mathbb{R}^2, V)$ have their standard linear structure, we end up with a linear structure that is always as least as fine as the old linear structure. We however want to start again from scratch, i.e. we want also to be able to define a linear structure that is less fine than the standard linear structures on $V \times V$ and $L(\mathbb{R}^2, V)$. To do this we must first get rid of the standard linear relations on $V \times V$ and $L(\mathbb{R}^2, V)$. This however is precisely what you do when you consider the free vector spaces $Span_{\mathbb{R}}(V \times V)$ and $Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$. That is the reason we will consider equivalence relations on these vector spaces. Furthermore note that while $V \times V$ with its standard linear structure is two-dimensional, $Span_{\mathbb{R}}(V \times V)$ is $|V|^2$-dimensional where $|V|$ is the cardinality of V as a set. The same holds for $Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$. Therefore you should always be cautious to not confuse the new linear structure with the standard one when performing operations on these spaces.

Theorem. *Let V be an n- dimensional vector space (with $n < \infty$) over $\mathbb{R}$. Define the subset U of $Span_{\mathbb{R}}(V \times V)$ as:*

$$U = \left\{ \begin{array}{c} \lambda(v,w) - (\lambda v, w) \\ \lambda(v,w) - (v, \lambda w) \\ (v + v', w) - (v,w) - (v', w) \\ (v, w' + w) - (v,w) - (v,w') \\ (v,v) \end{array} \middle| v, v', w, w' \in V \text{ and } \lambda \in \mathbb{R} \right\}$$

Then $\Lambda^2 V$ is isomorphic to $V \wedge V := Span_{\mathbb{R}}(V \times V)/Span(U)$ as a vector space.

First notice that $Span(U)$ is the normal span of U, which means that $Span(U)$ is the subspace generated by the elements of U and the linear structure of $Span_{\mathbb{R}}(V \times V)$. In particular this means that $Span(U)$ is by construction a linear subspace of $Span_{\mathbb{R}}(V \times V)$ and the quotient in $V \wedge V$ is well

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defined. The proof I will present here follows the path I set forth during the end of the work group. This may not be obvious at first sight. However, the second part of the proof that considers itself with proving $A \sim B$ if and only if $A - B \in Span(U)$ amounts to proving that the map I defined in the work group is independent of the representative.

Proof. The vector space $\Lambda^2 V$ is defined as $\Lambda^2 V := Span_{\mathbb{R}}(L(\mathbb{R}^2, V))/\sim$ where $\sim$ is the equivalence relation on $L(\mathbb{R}^2, V)$ given by $A \sim B$ if and only $\det(F \circ A) = \det(F \circ B)$ for all $F \in L(V, \mathbb{R}^2)$. By Lemma 11.1 this is equivalent to both A and B having rank smaller than 2 or A and B having the same image while there is a $G \in L(\mathbb{R}^2, \mathbb{R}^2)$ with $\det(G) = 1$ such that $A = BG$. To start the proof note that the map $\hat{f}\colon V \times V \to L(\mathbb{R}^2, V)$ given by

$$\hat{f}((v_1, v_2)) = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} =: \begin{pmatrix} v_1 & v_2 \end{pmatrix}$$

where $\begin{pmatrix} v_1 & v_2 \end{pmatrix} := \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ is the linear map whose matrix consists of the vectors v_1 and v_2 as columns, is a bijection. This follows immediately by picking bases. $\hat{f}$ can now be extended to a linear isomorphism $f\colon Span_{\mathbb{R}}(V \times V) \to Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$ by defining it to respect the linear structure of $Span_{\mathbb{F}}(V \times V)$, i.e. for $(a, b), (c, d) \in V \times V$ and $\lambda, \mu \in \mathbb{R}$ it holds:

$$f(\lambda(a, b) + \mu(c, d)) = \lambda\hat{f}((a, b)) + \mu\hat{f}((c, d))$$

This means that $Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$ and $Span_{\mathbb{R}}(V \times V)$ are isomophic. Now the proof will consist of showing that under this identification $A \sim B$ holds if and only if $A - B \in Span(U)$ holds.

Remember that the determinant map $\det\colon Mat_{\mathbb{F}}(n, n) \to \mathbb{F}$, where $Mat_{\mathbb{F}}(n, n)$ is the set of all $n \times n$ matrices with coefficients in the field $\mathbb{F}$, is the unique map from $Mat_{\mathbb{F}}(n, n)$ to $\mathbb{F}$ such that it is multilinear and alternating in the columns of the matrix and it holds $\det(I_n) = 1$ (see for example Chapter 10 in the lecture notes of Lineaire Algebra 1.) A map is called multilinear if it is linear in each of its components seperately. Note the difference between linearity and multilinearity. If $g\colon \mathbb{R}^2 \to \mathbb{R}$ is linear it holds:

$$\lambda g(a, b) = g(\lambda a, \lambda b)$$
$$g(a + c, b + d) = g(a, b) + g(c, d)$$

but if g is multilinear it holds:

$$\lambda g(a, b) = g(\lambda a, b) = g(a, \lambda b)$$
$$g(a + c, b + d) = g(a, b + d) + g(c, b + d) = g(a, b) + g(a, d) + g(c, b) + g(c, d)$$

A map is called alternating if interchanging two of its entries amounts to changing the sign in the outcome, so if g were alternating we'd have $g(a, b) = -g(b, a)$. Note that this definition still holds for maps with more than two entries.

Now let $A, B \in L(\mathbb{R}^2, V)$ such that $A \sim B$. Suppose that $A = 0$. Then $0 \sim B$ if and only if $rk(B) < 2$ by Lemma 11.1. Since $rk(B) < 2$ there exists a basis of V such that B can be written as $B = (B_1, 0)$ with $0, B_1 \in V$. Note that, since $(0 \cdot v, 0) = (0, 0) \in U$, it holds that $0 \cdot (v, 0) \in Span(U)$ and from this it follows that $(v, 0) \in Span(U)$ for all $v \in V$. Similarly we have $(0, v) \in Span(U)$ for all $v \in V$. Hence we have $0 - B = (0, 0) - (B_1, 0) \in Span(U)$.

Now let $rk(A) = 2$. Then by Lemma 11.1 we must also have $rk(B) = 2$, $Im(A) = Im(B)$ and there exists a $G \in L(\mathbb{R}^2, \mathbb{R}^2)$ with $\det(G) = 1$ such that $A = BG$. Fix bases in $\mathbb{R}^2$ and V and write $A = (A_1, A_2)$ and $B = (B_1, B_2)$ with $A_1, A_2, B_1, B_2 \in V$ and $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$. We thus have:

$$(A_1, A_2) = (B_1, B_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aB_1 + cB_2, bB_1 + dB_2)$$

Notice that elements of the form $(v, w) + (w, v)$ for $v, w \in V$ are also contained in $Span(U)$ because we have modulo elements in $Span(U)$:

$$\begin{aligned}
Span(U) \ni (v + w, v + w) &= (v + w, v + w) - ((v + w, v + w) - (v, v + w) - (w, v + w)) \\
&= (v, v + w) + (w, v + w) \\
&= (v, v + w) - ((v, v + w) - (v, v) - (v, w)) + (w, v + w) \\
&= (v, v) + (v, w) + (w, v + w) \\
&= (v, v) + (v, w) + (w, v + w) - ((w, v + w) - (w, v) - (w, w)) \\
&= (v, v) + (v, w) + (w, v) + (w, w) \\
&= (v, w) + (w, v)
\end{aligned}$$

(Looking at this equation modulo elements of $Span(U)$ means that an =-sign means that the right is contained in $Span(U)$ if the left is and vice versa.) This gives us modulo elements in $Span(U)$:

$$\begin{aligned}
A - B &= (aB_1 + cB_2, bB_1 + dB_2) - (B_1, B_2) \\
&= (aB_1 + cB_2, bB_1 + dB_2) - ((aB_1 + cB_2, bB_1 + dB_2) \\
&\quad - (aB_1, bB_1 + dB_2) - (cB_2, bB_1 + dB_2)) - (B_1, B_2) \\
&= (aB_1, bB_1 + dB_2) + (cB_2, bB_1 + dB_2) - (B_1, B_2) \\
&= (aB_1, bB_1 + dB_2) - ((aB_1, bB_1 + dB_2) - (aB_1, bB_1) \\
&\quad - (aB_1, dB_2)) + (cB_2, bB_1 + dB_2) - (B_1, B_2) \\
&= (aB_1, bB_1) + (aB_1, dB_2) + (cB_2, bB_1 + dB_2) - (B_1, B_2) \\
&= (aB_1, bB_1) + (aB_1, dB_2) + (cB_2, bB_1 + dB_2) \\
&\quad - ((cB_2, bB_1 + dB_2) - (cB_2, bB_1) - (cB_2, dB_2)) - (B_1, B_2) \\
&= (aB_1, bB_1) + (aB_1, dB_2) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= (aB_1, bB_1) - ((aB_1, bB_1) - a(B_1, bB_1)) + (aB_1, dB_2) \\
&\quad - ((aB_1, dB_2) - a(B_1, dB_2)) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= a(B_1, bB_1) + a(B_1, dB_2) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= a((B_1, bB_1) - ((B_1, bB_1) - b(B_1, B_1))) + a((B_1, dB_2) \\
&\quad - ((B_1, dB_2) - d(B_1, B_2))) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= ab(B_1, B_1) + ad(B_1, B_2) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= ad(B_1, B_2) + (cB_2, bB_1) + (cB_2, dB_2) - (B_1, B_2) \\
&= \cdots = ad(B_1, B_2) + bc(B_2, B_1) - (B_1, B_2) \\
&= ad(B_1, B_2) + bc((B_2, B_1) - ((B_2, B_1) + (B_2, B_1))) - (B_1, B_2) \\
&= ad(B_1, B_2) - bc(B_1, B_2) - (B_1, B_2) \\
&= (ad - bc - 1)(B_1, B_2) \\
&= (\det(G) - 1)(B_1, B_2) \\
&= 0 \cdot (B_1, B_2) \\
&\in Span(U)
\end{aligned}$$

since $0 \cdot (B_1, B_2)$ is the zero of $Span_{\mathbb{R}}(V \times V)$ and is thus contained in $Span(U)$.

We now conclude that if $A \sim B$ for $A, B \in L(\mathbb{R}^2, V)$ then $A - B \in Span(U)$. By linearity it now follows that if $A \sim B$ for $A, B \in Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$ then $A - B \in Span(U)$.

Now the other way around. Let $A = (A_1, A_2), B = (B_1, B_2) \in V \times V$ such that $A - B \in U$. Suppose there are $v, w \in V$ and $\lambda \in \mathbb{F}$ such that $A - B = \lambda(v, w) - (\lambda v, w)$. Now to calculate the determinant of $F \circ A$ we need to know how the determinant acts on formal linear combinations of linear maps. This is not predefined, but the natural way of defining it is by taking the linear extension of the standard determinant in the same way as we extended $\hat{f}$ to f at the beginning of the proof. Using this extension for the determinant we get by the properties of the determinant for all $F \in L(V, \mathbb{R}^2)$:

$$\begin{aligned}
\det(F \circ A) &= \det(F \circ (B + \lambda(v, w) - (\lambda v, w))) \\
&= \det(F \circ B + \lambda F \circ (v, w) - F \circ (\lambda v, w)) \\
&= \det(F \circ B) + \lambda \det(F \circ (v, w)) - \det(F \circ (\lambda v, w)) \\
&= \det(F \circ B) + \lambda \det((Fv, Fw)) - \det((\lambda Fv, Fw)) \\
&= \det(F \circ B) + \lambda \det((Fv, Fw)) - \lambda \det((Fv, Fw)) \\
&= \det(F \circ B)
\end{aligned}$$

So $A \sim B$. Similarly we can find $A \sim B$ if $A - B = \lambda(v, w) - (v, \lambda w)$. Now suppose there are $v, v', w \in V$ such that $A - B = (v + v', w) - (v, w) - (v', w)$. Then we have for all $F \in L(V, \mathbb{R}^2)$:

$$\begin{aligned}
\det(F \circ A) &= \det(F \circ (B + (v + v', w) - (v, w) - (v', w))) \\
&= \det(F \circ B + F \circ (v + v', w) - F \circ (v, w) - F \circ (v', w)) \\
&= \det(F \circ B) + \det(F \circ (v + v', w)) - \det(F \circ (v, w)) - \det(F \circ (v', w)) \\
&= \det(F \circ B) + \det((Fv + Fv', Fw)) - \det((Fv, Fw)) - \det((Fv', Fw)) \\
&= \det(F \circ B) + \det((Fv, Fw)) + \det((Fv', Fw)) - \det((Fv, Fw)) - \det((Fv', Fw)) \\
&= \det(F \circ B)
\end{aligned}$$

So $A \sim B$. Similarly we can find $A \sim B$ if $A - B = (v, w + w') - (v, w) - (v, w')$. Now suppose there is a $v \in V$ such that $A - B = (v, v)$. Then we have for all $F \in L(V, \mathbb{R}^2)$:

$$\begin{aligned}
\det(F \circ A) &= \det(F \circ (B + (v, v))) \\
&= \det(F \circ B + F \circ (v, v)) \\
&= \det(F \circ B) + \det(F \circ (v, v)) \\
&= \det(F \circ B) + \det((Fv, Fv)) \\
&= \det(F \circ B)
\end{aligned}$$

since (Fv, Fv) has rank at most 1 and is thus not an isomorphism. So $A \sim B$.

Hence we conclude that if $A, B \in V \times V$ such that $A - B \in U$ then $A \sim B$. By linearity it then follows that $A, B \in Span_{\mathbb{R}}(V \times V)$ such that $A - B \in Span(U)$ then $A \sim B$. Note that this works precisely because we have defined the determinant of a formal sum of linear maps to be the sum of the determinant of the linear maps, i.e. we have linearised the relation $\sim$ in the same way as we have linearised the relation of having a difference in U by considering the span of U.

We now conclude that under the isomorphism $f\colon Span_{\mathbb{R}}(V \times V) \to Span_{\mathbb{R}}(L(\mathbb{R}^2, V))$ the equivalence relations equal each other, i.e. $A \sim B$ if and only if $A - B \in Span(U)$. Hence $\Lambda^2 V := Span_{\mathbb{R}}(L(\mathbb{R}^2, V))/\sim$ and $V \wedge V := Span_{\mathbb{R}}(V \times V)/Span(U)$ are isomorphic as vector spaces. □

Some final remarks. The elements of $V \wedge V$ are of the form $v \wedge w := [(v, w)]$ with $v, w \in V$ where the brackets denote taking the equivalence class. As the computations above with elements modulo elements in $Span(U)$ have shown, the $\wedge$ (called a wedge or in Dutch 'wig') can also be viewed as an alternating bilinear product $\wedge\colon V \times V \to V \wedge V$ such that $V \wedge V = Span(\{v \wedge w | v, w \in V\})$. This is because the set U defining the equivalence relation says precisely that, if we take $v \wedge w = [(v,w)]$ as the definition of the wedge product, the $\wedge$ is bilinear and alternating. Because of the isomorphism all elements of $\Lambda^2 V$ are thus linear combinations of elements of the form $v \wedge w$ with $v, w \in V$ where we consider the $\wedge$ as a formal alternating bilinear map. This makes computations a lot easier. We can for example now easily find a basis for $\Lambda^2\mathbb{R}^3$. If (e_1, e_2, e_3) is the standard basis of $\mathbb{R}^3$, then the set $\{e_i \wedge e_j | i, j \in \{1,2,3\}\}$ certainly spans $\Lambda^2\mathbb{R}^3$. We however have the relations $e_i \wedge e_j = -e_j \wedge e_i$ and $e_i \wedge e_i = 0$. Hence $(e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$ is a basis of $\Lambda^2\mathbb{R}^3$. (That $v \wedge v = 0$ for all v follows from the fact that $v \wedge v = -v \wedge v$.)

Finally notice that the proof given here can be generalised to $\Lambda^k V$ being isomorphic to $Span_{\mathbb{R}}(V^k)/Span(U)$ for any $k \in \mathbb{N}$, but with U appropriately defined. In this way we get that the elements of $\Lambda^k V$ look like linear combinations of $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ with $v_1, \ldots, v_k \in V$ where $\wedge\colon V \times V \to V \wedge V$ is the same alternating bilinear product as above.