Practice Exam Manifolds 1 2018-2019

1. (A peculiar function) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + x^3$.
   (a) Prove that f is differentiable and calculate $f'(x)$ for $x \in \mathbb{R}$.
   (b) Prove that f is a diffeomorphism.

2. (A quarter of a torus) Let $S^{1^+} = \{(x, y) \in S^1 \mid x, y \geq 0\}$, and let $S^{1^{+'}} = \{(x, y) \in S^1 \mid x, y > 0\}$.
   (a) Prove that $S^{1^{+'}}$ and S^1 are C^1-manifolds.
   (b) Prove that $S^{1^{+'}} \times S^1$ is a C^1-manifold.
   (c) Find a basis for $\Lambda^2(\mathbb{R}^4)^*$.
   (d) Calculate $\int_{S^{1^+} \times S^1} (1 + x^4) dx^1 \wedge dx^3 - x^1 dx^3 \wedge dx^4$.

3. (Isometries in the hyperbolic plane) Consider the upper half-plane $\mathbb{H} \subset \mathbb{C} \cong \mathbb{R}^2$ with the hyperbolic metric. Show that all maps $\phi : \mathbb{H} \to \mathbb{H}$, $\phi(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ are isometries of $\mathbb{H}$.

4. (On sections and immersions) Let M be an m-dimensional smooth manifold and N be an n-dimensional smooth manifold, and let $f : M \to N$ differentiable. We call f an *immersion* if the function $f'(p) : \mathbb{R}^m \to \mathbb{R}^n, f'(p)(v) = \pi_2 \circ f'(p, v)$ is injective for all $p \in M$. (Here π_2 is the projection on the fibre of TN.)
   Now let M be a smooth manifold, let E be a smooth vector bundle over M, and let $s : M \to E$ be a smooth section. Prove that s is an immersion.