Exercises Chapter 5 Manifolds 1 2018-2019

1. Let $\alpha_i \in \mathcal{A}$ $(i \in \mathbb{N})$. Assume that for all $\alpha, \beta, \gamma \in \mathcal{A}$, we have transition functions τ_α^β such that $\tau_\alpha^\alpha = \text{id}$ and $\tau_\alpha^\gamma \tau_\gamma^\beta \tau_\beta^\alpha = \text{id}$.
   (a) Prove that $(\tau_\alpha^\beta)^{-1} = \tau_\beta^\alpha$.
   (b) Prove that $\prod_{i=1}^n \tau_{\alpha_i}^{\alpha_{i+1}} = \text{id}$ if $\alpha_{n+1} = \alpha_1$ for all $n \in \mathbb{N}$.
2. Let V be a real vector space. For all $n \in \mathbb{N}$, prove that V^n is a manifold.
3. For all $n \in \mathbb{N}$, prove that S^n is a manifold.
4. For $n \in \mathbb{N}$, define $\mathbb{R}P^n = \mathbb{S}^n / \sim$, where $x \sim y$ if and only if $x = \pm y$. Prove that $\mathbb{R}P^n$ is a manifold.
5. Prove that for all $n \in \mathbb{N}$, every atlas of S^n has more than one chart.
6. Let $n \in \mathbb{N}$ and let M and N be n-manifolds.
   (a) Assume that $M \cap N = \emptyset$. Prove that $M \cup N$ is a n-manifold.
   (b) Find M, N such that $M \cap N$ is not a manifold (of any dimension).
7. Let M, N be two C^k-manifolds. Prove that $M \times N$ is also a C^k-manifold.
8. Let $n \in \mathbb{N}$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ a C^k-function, and $c \in \mathbb{R}$. Then $S := f^{-1}(c)$ is called an *n-surface* if $df(p) \neq 0$ for all $p \in S$. Prove that S is a C^k-manifold.
9. Let X and Y be topological spaces, and let $\sim_X$ and $\sim_Y$ be equivalence relations on X and Y, with $q_X : X \to X/\sim_X$ and $q_Y : Y \to Y/\sim_Y$ the quotient maps, and let $f : X \to Y$ be a continuous map. Assume that for all $x_1, x_2 \in X$ with $x_1 \sim_X x_2$, we have that $f(x_1) = f(x_2)$.
   (a) Prove that there exists an unique continuous map $\tilde{f} : X/\sim_X \to Y$ such that $\tilde{f} \circ q_X = f$.
   (b) Prove that $q_Y \circ \tilde{f} : X/\sim_X \to Y/\sim_Y$ is continuous.
   (c) If $q_Y \circ \tilde{f} : X/\sim_X \to Y/\sim_Y$ is continuous, prove or disprove that f is also continuous.
10. Let X, Y be topological spaces, let X be compact and Y Hausdorff, and let $f : X \to Y$ be a continuous bijection. Prove that f is a homeomorphism.
11. Let $n \in \mathbb{N}$. Prove that $SO(n) := \{A \in O(\mathbb{R}^n) \mid \det(A) = 1\}$ is manifold.
12. Let $n \in \mathbb{N}$. Prove that $\text{Sym}(n) := \{A \in L(\mathbb{R}^n, \mathbb{R}^n) \mid A = A^\top\}$ is manifold.

13. (Traditional definition of manifold)
Exercise 5.1.5 from the syllabus.
In the literature manifolds are usually defined in a slightly different way as follows. To make the temporary distinction we call them T-manifolds. An m-dimensional C^k T-manifold is a second countable Hausdorff space M together with for each $p \in M$ a homeomorphism $p \in U \xrightarrow{\phi} V \subset \mathbb{R}^m$ between open subsets such that for any two such homeomorphisms ϕ, ψ the map $\phi \circ \psi^{-1}$ is a C^k-diffeomorphism.
In this exercise you will check that the two definitions are really equivalent. To build an atlas from a T-manifold we take the charts to be the open sets $V \subset \mathbb{R}^m$ that are the target of the homeomorphisms ϕ. The transition maps are just the $\phi \circ \psi^{-1}$. In the other direction, given an atlas, Lemma 22 gives us homeomorphisms $M^\alpha \to q(M^\alpha) \subset M$. Their inverses are the ϕ we are looking for in the definition of T-manifold.

14. Let $\mathcal{B}$ be the atlas defined by the holomorphic pairs (U_α, f_α) with $\alpha \in \mathcal{A}$, as in Definition 34. Prove that the quotient space induced by this atlas is second countable and Hausdorff, thus a manifold.

15. (a) Calculate the quotient space M of the atlas of the logarithm, as explained in section 5.3.
    (b) Draw a picture of M.

16. (The cylinder)
Let $\mathbb{T}$ be the circle obtained by the two charts $\mathbb{T}_0 = \mathbb{T}_1 = (-1, 1)$ and transition functions $\tau_0^1(t) = \tau_1^0(t) = \begin{cases} t-1 & \text{if } t > 0 \\ t+1 & \text{if } t < 0 \end{cases}$.
Let E be a vector bundle over M with fiber $\mathbb{R}$ with atlas (E_0, E_1), where $E_0 = \mathbb{T}_0 \times \mathbb{R}$ and $E_1 = \mathbb{T}_1 \times \mathbb{R}$, and where $\epsilon_0^1 = \epsilon_1^0 = (\tau_0^1, g_0^1)$, where $g_0^1|_{(-1,0)\cup(0,1)} = 1$.
    (a) Prove that E is diffeomorphic to $S^1 \times \mathbb{R}$.
    (b) Prove that E has a section which is nowhere zero.

17. (The Möbius band)
Let $\mathbb{T}$ be the circle obtained by the two charts $\mathbb{T}_0 = \mathbb{T}_1 = (-1, 1)$ and transition functions $\tau_0^1(t) = \tau_1^0(t) = \begin{cases} t-1 & \text{if } t > 0 \\ t+1 & \text{if } t < 0 \end{cases}$.
Let E be a vector bundle over M with fiber $\mathbb{R}$ with atlas (E_0, E_1), where $E_0 = \mathbb{T}_0 \times \mathbb{R}$ and $E_1 = \mathbb{T}_1 \times \mathbb{R}$, and where $\epsilon_0^1 = \epsilon_1^0 = (\tau_0^1, g_0^1)$, where $g_0^1|_{(-1,0)} = 1$ and $g_0^1|_{(0,1)} = -1$.
    (a) Prove that the transition functions ϵ_α^β for $\alpha, \beta \in \{0, 1\}$ satisfy the cocycle condition.
    (b) Prove that E is diffeomorphic to the Möbius band.

(c) Prove that E does not have a section which is nowhere zero.

18. Let $k \in \mathbb{N}$, and let E be a vector bundle over a manifold M with fiber $\mathbb{R}^k$. Let $\epsilon_\alpha^\beta = (\tau_\alpha \alpha^\beta, L_\alpha^\beta)$ be the transition functions of E, with $L_\alpha^\beta \in GL(\mathbb{R}^k)$, $\alpha, \beta \in \mathcal{A}$. Show that $\Lambda^k E$ is a vector bundle over M with fiber $\Lambda^k \mathbb{R}^k \cong \mathbb{R}$, with transition functions ${\epsilon'}_\alpha^\beta = (\tau_\alpha^\beta, \det(L_\alpha^\beta))$.

19. (Tangent space) Let M be an n-dimensional smooth manifold, and $p \in M$. A smooth curve in M through p is a smooth map $\gamma : (-\varepsilon, \varepsilon) \to M$ ($\varepsilon \in \mathbb{R}_{>0}$) such that $\gamma(0) = p$. Denote the space of such curves by $\mathcal{K}_p$. Let $\alpha \in \mathcal{A}$ and $q : M^\alpha \to M$ be a coordinate patch such that $p \in q(M^\alpha)$.
Define $d_q(\gamma) = (q^{-1} \circ \gamma)'(0) \in \mathbb{R}^n$.
Define $\sim$ on $\mathcal{K}_p$ by $\gamma_1 \sim \gamma_2 \Leftrightarrow d_q(\gamma_1) = d_q(\gamma_2)$.

    (a) Prove that $\sim$ is independent from the choice of coordinate patch.

    Define $T_pM := \mathcal{K}_p / \sim$.

    (b) Prove that T_pM is an n-dimensional real vector space.

20. Let M be a manifold. Prove that $TM = \bigsqcup_{p \in M} T_pM$.

21. Find a nowhere vanishing C^1 1-covector field on S^1.

22. Find a C^1 1-vector field on S^2 with

    (a) 2 zeros;
    (b) 1 zero.

23. Find a (non-zero) C^1 2-vector field on $M = f^{-1}(0)$ with
$f : \mathbb{R}^3 \to \mathbb{R},\ (x, y, z) \mapsto x^2 + y^2 - 1 - z^2$.

24. Choose $n \in \mathbb{N}_{\geq 1}$. Find a (non-zero) C^1 2-vector field on S^2 with n zeros.

25. Prove that a section is an immersion (that is, prove that the derivative of a section is injective.)

26. One important (and nice) theorem on differential manifolds is the theorem of Gauss-Bonnet, which states that for a $2n$-dimensional compact connected oriented manifold M ($n \in \mathbb{N}$), we have that $\int_M \kappa = 2\pi\chi(M)$, where $\kappa(p)$ is the curvature of M at p, and $\chi(M)$ the Euler characteristic of M

    (a) Let $m \in \mathbb{N}$. Prove that there exists a $p \in S^m$ such that $\kappa(p) > 0$.

    By symmetry we can then assume that $\kappa(p) > 0$ for all $p \in S^m$.
    Now let V be a 1-vector field on M such that $p_1, \ldots, p_r$ are the zeros of V. For $i \in \{1, \ldots, r\}$, let $p_i \in U_i \subset M$ be a neighbourhood of p_i such that U_i is homeomorphic to D^{2n} and such that $p_j \notin U_i$ for all $j \in \{1, \ldots, r\} \backslash \{i\}$. We define the *index* of V at p, $\iota_v(p_i)$, to be the degree of the map $V'|_{\partial U} : \partial U \to S^{2n-1}, V'|_{\partial U}(q) = V(q)/||V(q)||$. The Poincaré-Hopf theorem states that $\sum_{i=1}^r \iota_v(p_i) = \chi(M)$.

(b) Prove the Hairy Ball Theorem: there exists a nowhere vanishing 1-covector field on S^n if and only if n is odd.

27. Let E be a vector bundle over a manifold M, and let $s_1, \ldots, s_n$ $(n \in \mathbb{N})$ be sections of E such that for all $p \in M$, the elements $(s_i(p))|_{i=1}^n$ form a basis of E_p. Prove that E is isomorphic to $M \times \mathbb{R}^n$.

28. Let M be an n-dimensional manifold, and let $v \in \mathbb{R}^n \setminus \{0\}$. By identifying T_pM and $\mathbb{R}^n$ for all $p \in M$, prove that the musical isomorphism $g^\flat : TM \to TM^*, g^\flat(p)(w) = g(p)(v, w)$ is indeed an isomorphism.