Exercises Chapter 4 Manifolds 1 2018-2019

1. Let V be a real vector space.

   (a) Prove that
   $$O(V) = \{M \in L(V,V) \mid MM^\top = I\} = \{M \in L(V,V) \mid \det(M) = \pm 1\}.$$

   (b) Prove for $M \in O(V)$:

      i. $\det(M) = 1$ if and only if M is a rotation;

      ii. $\det(M) = -1$ if and only if M is a reflection.

2. Let V be a real vector space. Prove that $O(V)$ is a group.

3. Let $V = \mathbb{R}^2$. Prove for $\star : V \to V$ that $\star^2 = -\text{id}$.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ is smooth, and let F be a 1-covector field on $\mathbb{R}^3$. Prove that

   (a) $\text{grad}(f) = df$;

   (b) $\text{curl}(F) = \star dF$;

   (c) $\text{div}(F) = \star d \star F$;

   (d) $\Delta f = \star d \star df$.

5. Can you retrieve the definition of a metric from Topology from our definition? If yes, how? If no, why not?

6. Calculate $L(\gamma)$ twice, once using the Euclidean metric and once using the hyperbolic metric in the cases that

   (a) $\gamma : [0,1] \to \mathbb{H},\ \gamma(t) = (0, 1+t)$.

   (b) $\gamma : [0,1] \to \mathbb{H},\ \gamma(t) = (\sin(\pi t), \cos(\pi t) + a)$ for $a \geq 1$;

   (c) $\gamma : [0,1] \to \mathbb{H},\ \gamma(t) = (\sin(4\pi t), 2 + \cos(2\pi t))$.

7. Show that all maps $\phi : \mathbb{H} \to \mathbb{H},\ f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$ are isometries of $\mathbb{H}$.

8. Let $P, Q \subset \mathbb{R}^n$ be open ($n \in \mathbb{N}$), and let g_P be a metric on P, and g_Q a metric on Q. Let $p_0, p_1 \in P$, and let $\phi : P \to Q$ be an isometry. Prove that the path $\gamma : [0,1] \to P$ from p_0 to p_1 has minimal length if and only if $\phi \circ \gamma$ has minimal length.

9. (a) Let $a, b \in \mathbb{R}_{>0}$ and let $x \in \mathbb{R}$. Prove that the path with the shortest length from (x, a) to (x, b) is on the vertical line between those points.

   (b) Prove that all half circles can be obtained as images of vertical lines of isometries of $\mathbb{H}$.

(c) Let $a_1, a_2, a_3 \in \mathbb{H}$, and $b_1, b_2, b_3 \in \mathbb{H}$ be two sets of distinct points. Prove that there exists an isometry $\phi : \mathbb{H} \to \mathbb{H}$ such that $\phi(a_i) = b_i$ for all $i \in \{1, 2, 3\}$.

(d) Let $x_1, x_2, x_3 \in \mathbb{R}$ be distinct points, and let T_n be triangle with vertices $(x_1, \frac{1}{n}), (x_2, \frac{1}{n}), (x_3, \frac{1}{n})$. Calculate the area of $T = \lim_{n\to\infty} T_n$. (Hint: consider the triangle with "improper" vertices at $(0, 0), (1, 0)$ and $y = \infty$.)

10. Let $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ be 1-vector fields on $\mathbb{R}^3$. Now define $F = E + B$, where

$$\begin{aligned} E &= E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz, \text{and} \\ B &= B_x dz \wedge dy + B_y dx \wedge dz + B_z dy \wedge dx. \end{aligned}$$

Furthermore, for $p = (t, x, y, z) \in \mathbb{R}^4$ and $v, w \in \mathbb{R}^4$, let $g_{Max} : \mathbb{R}^4 \to \text{Bil}(\mathbb{R}^4, \mathbb{R}^4)$ given by

$$g_{Max}(p)(v, w) = v^\top \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} w.$$

We call two vectors $v, w \in \mathbb{R}^4$ orthonormal if $g_{Max}(p)(v, w) = 0$ for $p \in \mathbb{R}^4$. (So pretend that g_{Max} is a metric.)
Prove that we can rewrite the Maxwell equations from Exercise 3.3 in vacuum (so $\rho = 0$ and $\vec{J} = 0$) to

$$dF = 0, \qquad \star d \star F = 0.$$