Exercises Chapter 2 Manifolds 1 2018-2019

1. (a) Find a point $p = (x, y) \in \mathbb{R}^2$ satisfying the equation $f(x, y) = \cos^2(x) + y^3 - 1 = 0$.
   (b) Find the first order Taylor expansion of f at p in both x and y (i.e. linearise f).
   (c) Solve the linear equation.
   (d) If possible, solve $f(x, y) = 0$ and compare this to the solution of the linear equation.
2. The same as exercise 1, but for $f(x, y) = \cos^2(x) + \tan(y) = 0$.
3. The same as exercise 1, but for $f(x, y) = \cos^2(x) + \sin^2(y) = 0$.
4. The same as exercise 1, but for the system
$$f(x, y, z) = x + y^2 + z - 1 = 0, \; g(x, y, z) = y^2 + z^2 - 1 = 0.$$
5. Prove that for any $n \times n$-matrix, we can interchange two rows and/or columns by multiplying only by the matrices R^c_{ij}.
6. Put $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -3 & 1 & 0 \end{pmatrix}$ in Gaussian eliminated form.
7. Prove or disprove: if $A = ED\tilde{E}$ is written in Gaussian eliminated form, then D has only eigenvalues of A on the diagonal.
8. Prove: two symmetric square matrices A, A' of the same size have the same number of positive, negative and zero eigenvalues if and only if there exists an invertible matrix C such that $A = CAC^\top$.
9. Let V be a real vector space. Then we define $V \wedge V := (V \times V)/U$, where $V \times V := \text{span}\{(v, w) \mid v, w \in V\}$ and
$$U = \text{span}\left\{ \begin{matrix} (\lambda \cdot v, w) - (v, \lambda \cdot w) \\ (v + v', w) - (v, w) - (v', w) \\ (v, w + w') - (v, w) - (v, w') \\ (v, v) \end{matrix} \;\middle|\; v, v', w, w' \in V, \lambda \in \mathbb{R} \right\}.$$
For $v, w \in V$,we write $v \wedge w \in V \wedge V$. Find a basis for $\mathbb{R}^2 \wedge \mathbb{R}^2$ and for $\mathbb{R}^3 \wedge \mathbb{R}^3$.
10. Let $f, g : \mathbb{R}^m \to \mathbb{R}^n$ and $\alpha : \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions, and let $t \in \mathbb{R}$. Prove or disprove the following statements:
    (a) If $f = o(h) = g$, then $f + g = o(h)$.
    (b) If $f = o(h)$, then $t \cdot f = o(h)$.

(c) If $f = o(h) = g$, then $\langle f, g \rangle = o(h)$.

(d) If $f = o(h)$, then $\alpha \circ f = o(h)$.

11. Prove: if there are two linear maps $A, B : \mathbb{R}^m \to \mathbb{R}^n$ such that $Ah - Bh = o(h)$ $(h \in \mathbb{R}^m)$, then $A = B$.

12. (a) Let $\mathbb{R}^m \overset{open}{\supset} P \xrightarrow{f} \mathbb{R}$ be differentiable. Prove that $f' = (\nabla f)^\top$.

    (b) In the one-dimensional case, does our definition of differentiability equal that of Analysis 1?

13. (a) Prove that differentiable functions are continuous.

    (b) Prove that derivatives are continuous.

14. Prove that $L(\mathbb{R}^n, \mathbb{R}^n) \ni A \overset{exp}{\mapsto} e^A \in L(\mathbb{R}^n, \mathbb{R}^n)$ is differentiable at $A = 0$.

15. Using the canonical isomorphism $\mathbb{C} \simeq \mathbb{R}^2$, prove that $S^1 \ni z \mapsto z^n \in \mathbb{C}$ is differentiable for all $n \in \mathbb{N}$.

16. Pick a point $p \in S^2$. Find a diffeomorphism $F : S^2 \setminus \{p\} \to \mathbb{R}^2$.

17. Prove that the derivative of a diffeomorphism is an isomorphism of vector spaces.

18. Let $X \subset \mathbb{R}^n$ be open. Prove that the set of diffeomorphisms from X to itself form a group.

19. Find a bijection from $\mathbb{R}$ to itself that is C^1 but not a diffeomorphism.

20. For a C^2 map $f : \mathbb{R}^m \to \mathbb{R}^n$ and two vectors $v_1, v_2 \in \mathbb{R}^m$, prove that

$$\partial_{v_1} (\partial_{v_2} f) = \partial_{v_2} (\partial_{v_1} f) \,.$$

21. Find a differentiable $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that there is a line segment from a to b such that for all points c on that line segment, we have that $F(b) - F(a) \neq F'(c)(b - a)$.

22. Prove that $\mathbb{R} \ni x \overset{f}{\mapsto} x + x^3 \in \mathbb{R}$ is a diffeomorphism.

23. Show that $\mathbb{R}^2 \ni (x, y) \overset{f}{\mapsto} e^x(\cos y, \sin y) \in \mathbb{R}^2$ satisfies $f' \neq 0$ everywhere, but that is not a bijection and hence does not have a (global) inverse.

24. (a) Apply Newton's method to

$$\begin{aligned} G : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} e^{x+y} \\ y \log(1 + x^2) \end{pmatrix} \end{aligned}$$

    to find (or approximate) a pair $(x, y) \in \mathbb{R}^2$ solving $(e, 1) = G(x, y)$.

(b) Is this pair unique?

25. Let $A \in L(\mathbb{R}^m, \mathbb{R}^n)$. Prove that $\mathrm{rk}(A) \geq k$ if and only if there exists a $k \times k$-invertible submatrix of A.

26. Given a C^∞ map $f : \mathbb{R}^M \overset{open}{\supset} P \to \mathbb{R}^n$ such that $f'(p)$ is injective for $p \in P$, prove that there are homeomorphisms $\phi_1 : \mathbb{R}^m \overset{open}{\supset} U_1 \to V_1 \overset{open}{\subset} \mathbb{R}^m$ and $\phi_2 : \mathbb{R}^n \overset{open}{\supset} U_2 \to V_2 \overset{open}{\subset} \mathbb{R}^n$ such that

    i. $0 \in U_1$ and $\phi_1(0) = p$;
    ii. $0 \in U_2$ and $\phi_2(0) = f(p)$;
    iii. $U_1 \times \{0\} = U_2 \cap (\mathbb{R}^m \times \{0\})$;
    iv. $f(V_1) = V_2 \cap f(P)$, and
    v. $\phi_2^{-1} f \phi_1(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_m, 0 \dots, 0)$.

    You can use the following steps:

    (a) Find two maps $\phi_{1,2}$ satisfying properties i. and ii.

    (b) Find a suitable variant of the map

    $$G : U_1 \times \mathbb{R}^{n-m} \to \mathbb{R}^n, G(x, z) = \phi_2^{-1} f \phi_1(x) + (0, z)$$

    for $x \in U_1, z \in \mathbb{R}^{n-m}$ and apply the Inverse Function Theorem.

27. Using the canonical isomorphism $\mathbb{C} \simeq \mathbb{R}^2$, prove that holomorphic functions are differentiable. Under what conditions can differentiable functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ be seen as holomorphic functions?

28. (a) Pick $b \in \mathbb{R}^3 \setminus \{0\}$. Prove that $M_b : \mathbb{R}^3 \to \mathbb{R}^3, a \mapsto a \times b + \langle a, b \rangle b$ is a diffeomorphism.

    (b) Let $H \subset \mathbb{R}^n$ be a hyperplane. Prove that the reflection through H is a diffeomorphism.

29. Prove that

    $$F : \mathbb{R}^2 \to \mathbb{R}^2$$
    $$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \arctan(\sqrt{17e}x) \\ x^2y^3 + e^y \end{pmatrix}$$

    is a diffeomorphism.

30. Let $f, g_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be C^1 functions for $1 \leq i \leq m$, $(m \leq n)$ such that for all $1 \leq i \leq m$ and $x \in \mathbb{R}^n$, $g_i(x) = 0$ implies that $g_i'(x) \neq 0$. Let $p \in \mathbb{R}^{n+1}$ be a point such that $g_i(p) = 0$ for all $1 \leq i \leq m$. Prove the following implication:

    If f has a local extremum in p, then $\{f'(p)^\top\} \cup \bigcup_{i=1}^m \{g_i'(p)^\top\}$ is a set of linear dependent vectors.

Hint: Assume the contrary. Then prove that there exists a curve $\gamma : (-\epsilon, \epsilon) \to X$ on $X = \{x \in \mathbb{R}^{n+1} \mid g_i(x) = 0 \text{ for all } 1 \leq i \leq m\}$ such that $\gamma(0) = p$ and such that $f(\gamma(-\delta)) < f(\gamma(0)) < f(\gamma(\delta))$ for some $\delta < \epsilon$.
If this is too complicated, it is also very instructive to only prove the implication for $n = 1$.