Solutions to selected exercises

Exercise 9 (Cone.) For $a > 0$ imagine a cone

$$C = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = a^2 x^2 \text{and } x > 0\}$$

a. Write down the geodesic equation on C explicitly.

To better understand the cone we unwrap it as follows. Define the unwrapped cone as $U_\alpha = \{p \in \mathbb{R}^2 : \arg(p) \in [0, \alpha), |p| \neq 0\}$ here arg means the angle in $[0, 2\pi)$ of vector p with respect to the positive x-axis. The map $W_\alpha : U_\alpha \to \mathbb{R}^3$ that does the wrapping is

$$W_\alpha(p) = |p|(\sqrt{1 - \left(\frac{\alpha}{2\pi}\right)^2}, \frac{\alpha}{2\pi} \cos\left(\arg(p)\frac{2\pi}{\alpha}\right), \frac{\alpha}{2\pi} \sin\left(\arg(p)\frac{2\pi}{\alpha}\right))$$

b. Compute the correct value of $\alpha \in [0, 2\pi]$ such that W_α actually maps into and onto C. In other words W_α is a bijection onto C.

c. Verify that W_α sends straight line segments in U_α to geodesics in C.

Bonus. For what values of α can you make a geodesic intersect itself?

d. For $q = (1, a, 0)$, check that $v = (q, q) \in C_q$, the tangent space of C at point q.

e. Given the smooth curve $\gamma : [0, 2\pi] \to C$ defined by $\gamma(t) = (1, a\cos(t), a\sin(t))$, Compute the parallel transport $P_\gamma(v)$.

Solution: For convenience we identify $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{C}$ and consider $C = f^{-1}(0)$ for $f(x, w) = |w|^2 - a^2 x$.

Part a. The geodesic equation is equation (G) bottom of page 41 of from chapter 7 of Thorpe. We choose the normal vector field $\mathbf{N} = \frac{\nabla f}{|\nabla f|}$ so we first compute for $p = (x, w)$ that $\nabla f(p) = (p, 2(-a^2 x, w))$ and $\mathbf{N}(p) = (p, \frac{(-a, \frac{w}{ax})}{\sqrt{a^2+1}})$. Since α is already in use we call the unknown geodesic β with $\beta(t) = (\beta_1(t), \beta_2(t))$, where β_1, β_2 are unknown functions (the first is real, the last complex valued). $\frac{d}{dt}\mathbf{N}(\beta(t))(p) = \frac{1}{\sqrt{a^2+1}}(p, (0, \left(\frac{\beta_2}{\beta_1}\right)'))$. It follows that the geodesic equation for β may be written as:

$$(\ddot{\beta_1}, \ddot{\beta_2}) + \frac{1}{a^2+1}\mathrm{Re}\left(\bar{\dot{\beta_2}}\left(\frac{\beta_2}{\beta_1}\right)'\right)(-a^2, \frac{\beta_2}{\beta_1}) = 0$$

The real part comes from writing the dot product in $\mathbb{C}$ as $z \cdot w = \mathrm{Re}(\bar{z}w)$.
Part b. The point of the wrapping map W_α is to place U_α into $\mathbb{R}^3$ without distorting distance. Intuitively this means that it will transfer geodesics and parallel transport, angles and areas in the plane onto the same notions in C. Of course we do need to find the right α for the fixed a determining our cone C. Simply plugging in W_α into f gives $(\frac{\alpha}{2\pi})^2 - a^2(1+(\frac{\alpha}{2\pi})^2) = 0$ so apparently $\alpha = \frac{2\pi a}{\sqrt{a^2+1}}$. More geometrically a is the slope of the line L in the (x, y) plane that was rotated around the x-axis to make the cone. α is the circumference of the circular section of C cut off at length 1 measured along L.
Part c. As before we make use of complex numbers to streamline our polar coordinates. We also identify U_α with a region in the complex plane and rewrite W_α as follows.

$$W(p) = \frac{|p|}{c}\left(\frac{1}{a}, \left(\frac{p}{|p|}\right)^c\right) \qquad c = \frac{\sqrt{a^2+1}}{a}$$

Here we first collected the cosine and sine into a complex exponential and then used $\log z = \log|z| + i \arg(z)$. We also substituted the value for α found in the previous part.
Now consider a straight line δ in U_α parameterized as $\delta(t) = u + mt$ with $u, m \in \mathbb{C}$ and $u \in U_\alpha$. We would like to check that the curve given by $\beta(t) = (\beta_1, \beta_2) = W(\delta(t))$ is a geodesic. This can be done by showing that β solves the geodesic equation, but in this case it is easier to just show that the acceleration is λ times the normal vector for some function λ. This means $\ddot{\beta}_1 = -\lambda a^2$ and so all we need to show is $\frac{\ddot{\beta}_2}{\ddot{\beta}_1} = -a^{-2}\frac{\beta_2}{\beta_1}$. It is helpful to first prove $\dot{\beta}_2 = \frac{\mathrm{Re}(\bar{\delta}m)+ic\mathrm{Im}(\bar{\delta}m)}{\delta\bar{\delta}}\beta_2$. This follows from $|z| = \sqrt{z\bar{z}}$ and $|\delta|' = \frac{\delta\bar{m}+\bar{\delta}m}{2|\delta|}$.

Bonus. A geodesic intersecting itself can be obtained from taking a line connecting two points on the two straight lines in the boundary of U_α. This can happen precisely when $\alpha < \pi$.

Part d. The vector $v = (q, q)$ is in the tangent space since $v \cdot \mathbf{N}(q) = -a^2 + a^2 = 0$.
Part e. The parallel transport of $P_\gamma(v)$ is defined to be $V(2\pi)$ where V is the unique parallel and tangent vector field along γ such that $V(0) = v$. We guess V by taking a constant vector field on $W^{-1}(\gamma)$ and bringing that back to $\mathbb{R}^3$ with W. In other words we unwrap the cone and take copies of the image of vector v and place them all around the circular arc $W^{-1}(\gamma)$. As W does not change angles or lengths we may reconstruct this vector field by remembering its angle with respect to rays coming from the tip of the cone.
Concretely we first build a basis for the tangent space C_p as $\mathbf{R}(p) = (p, (1, \frac{y}{x}, \frac{z}{x}))$ and $\mathbf{M}(p) = \mathbf{N} \times \mathbf{R} = (p, 0, \frac{z}{ax}, \frac{-y}{ax})$ and then write our guess (draw a picture!):

$$\mathbf{V}(t) = \sqrt{a^2+1}\left(\gamma(t), \cos\left(\frac{t}{c}\right)\mathbf{R}(\gamma(t)) + \sin\left(\frac{t}{c}\right)\mathbf{M}(\gamma(t))\right)$$

Notice that $\mathbf{V}(0) = v$ and differentiation shows that $\dot{\mathbf{V}}(t) = \frac{\sin\left(\frac{t}{c}\right)}{\sqrt{a^2+1}}\mathbf{N}(t)$. Therefore V is indeed parallel and the parallel transport is

$$P_\gamma(v) = \mathbf{V}(2\pi) = \left(q, \cos\left(\frac{2\pi}{c}\right), a\cos\left(\frac{2\pi}{c}\right), -\sqrt{a^2+1}\sin\left(\frac{2\pi}{c}\right)\right)$$

Perhaps more informatively this is the vector v but rotated in the tangent space C_q along angle α.