Overview Manifolds 1

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December 18, 2017

1 Introduction

This is a companion to the introductory course on manifolds and differential geometry based on the book *Elementary topics in differential geometry* by J. Thorpe. In the hope of attaining a cleaner presentation I rearranged the material as follows. First we deal with general techniques of analysis in $\mathbb{R}^n$. Next we introduce the various kinds of surfaces found in the book. Then we show how to lift the techniques we learned in the first part to the surfaces. Finally we apply everything we learned to prove some non-trivial theorems including the Gauss-Bonnet theorem.

2 Analysis in finite dimensional inner product vector spaces

In this section we take V to be a finite dimensional vector space, whose dimension is denote by n. Even though V is isomorphic to $\mathbb{R}^n$ it is a good habit not to use such an isomorphism lightly [1]. If necessary W will denote another vector space of dimension m. For convenience we will assume V, W also have an inner-product denoted $\cdot$ but actually much of the theory works without it.

A **basis** of V is a linear isomorphism $\beta : V \to \mathbb{R}^n$. Often β is specified by giving the vectors $\beta^{-1}(e_i)$ where e_i is the vector in $\mathbb{R}^n$ with all zeros except for a 1 in the i-th slot.

V becomes a topological space by picking a basis $\beta : V \to \mathbb{R}^n$ and requiring it to be a homeomorphism. Since change of basis is a linear map from $\mathbb{R}^n$ to itself, this topology does not depend on the choice of basis. For brevity A and B will always denote open subsets of V and W.

The **tangent space** of vectors at point $p \in V$, notation V_p is defined as $V_p = \{(p, v) | v \in V\}$. It is a vector space isomorphic to V via the isomorphism $(p, v) \mapsto v$. However in this course it is very important to make the distinction between V and V_p. Also notice we may consider V_p as a subset (not subspace!) of $V \times V$, namely $\{p\} \times V$. Often v will be used to denote an element of V_p so

[1] The first rule of Linear Algebra is: **never choose a basis**.

the point p may not always be apparent in the notation. When confusion may arise the notation $\mathbf{v} = (p, v)$ is used.

The **support** of a function $f : A \to B$ is the closure of the set where $f \neq 0$.

We now turn to the theory of differentiation of functions $f : A \to B$ where as said $A \subset V$ and $B \subset W$ are open subsets. First if $A \subset \mathbb{R}$ is an open interval then f is called a **parameterized curve** (**path** for short). In this case the domain (open interval) is often denoted I instead of A. Paths are supposed to correspond to our intuition of walking around in space and this is why we build everything starting from those. Unless stated otherwise we assume $0 \in I$ for convenience.

The **derivative of a path** $\alpha : I \to B$ is the function $D\alpha : I \to B$ given by $D\alpha(t) = \lim_{\epsilon \to 0} \frac{\alpha(t+\epsilon)-\alpha(t)}{\epsilon}$ provided the limit exists. Often the sloppy notation $\frac{d\alpha}{dt}(p)$ is used for $D\alpha(p)$. Higher derivatives such as $DD\alpha$ may or may not exist but for simplicity we will usually assume they do and are continuous too. Such paths are called **smooth**. More generally a function $f : A \to B$ is defined to be smooth if for all smooth paths $\alpha : I \to A$ the composition $f \circ \alpha$ is smooth. A function f defined on a general subset $C \subset V$ (C is not necessarily open) is said to be smooth if there is a smooth function $\tilde{f}$ such that its restriction to C coincides with f, so $f = \tilde{f}|_C$.

The **velocity vector of a path** (p.8) $\alpha : I \to B$ is a map $\dot{\alpha} : I \to B \times W$ defined by $\dot{\alpha}(t) = (\alpha(t), D\alpha(t))$. More generally, for a function $f : A \to B$ the **directional derivative** (p.54) with respect to a vector $v \in V_p$, notation $\nabla_v f$, is defined by $\nabla_v f = D(f \circ \alpha)(0)$ where $\alpha : I \to A$ is any smooth path such that $\dot{\alpha}(0) = v$. Often a basis $\beta : V \to \mathbb{R}^n$ is chosen and $\nabla_{\beta^{-1}(e_i)} f$ is called the i-th partial derivative (with respect to the basis). A common (but sloppy) notation is $\frac{\partial f}{\partial x_i}(p)$ where x_i is supposed to be the i-th coordinate. Again fixing the basis, the gradient ∇f of a function $f : A \to \mathbb{R}$ is the vector of its partial derivatives (with respect to β).

The **differential** of a smooth map $f : A \to B$ is denoted df. It is a map $df : A \times V \to B \times W$ defined by $df(v) = (f \circ \alpha)^{\cdot}(0)$ where α is any smooth path with $\dot{\alpha}(0) = v$. For $a \in A$ the restriction of df to V_a is denoted $df_a : V_a \to W_{f(a)}$. It is a linear map. The matrix of df with respect to a pair of bases of V and W is just the Jacobian matrix of partial derivatives of f.

Theorem 1. *Chain rule (p.117)*
Given smooth $f : A \to B$ and $g : B \to C$ with C an open subset of a vector space we have: $d(g \circ f) = dg \circ df$

The set $T(A) = A \times V = \bigcup_{p \in A} V_p$ is often called the **tangent bundle** of A and is then denoted $T(A)$. Notice we may write $df : T(A) \to T(B)$.

A map $f : A \to B$ is called a **diffeomorphism** if it is a smooth bijection whose inverse is smooth too. We say that f is a local diffeomorphism around a if there is an open $a \in A' \subset A$ such that $f|_{A'}$ is a diffeomorphism.

Theorem 2. *Inverse function theorem (p.121)*
A smooth map $f : A \to B$ is local diffeomorphism around $a \in A$ if and only if df_a is a vector space isomorphism.

A (smooth) **partition of unity** subordinate to a finite family of open subsets $\{A_i\} \subset V$ is a family of (smooth) functions $\pi_i : V \to [0,1]$ such that: the support of π_i is inside A_i and $1 = \sum_i \pi_i$ (hence the name). For any compact subset $C \subset V$ and any open covering $\{A_i\}$ of C a partition of unity subordinate to it exists.

2.1 Vector fields and differential forms

A **vector field** on A is a function $\mathbf{X} : A \to V \times V$ such that $X(p) \in V_p$ for all p. There is a function $X : A \to V$ associated to each vector field, namely the function such that $\mathbf{X}(p) = (p, X(p))$. A vector field is smooth if the associated function is.

Given a basis $\beta : V \to \mathbb{R}^n$ we consider the **coordinate vector fields** $\mathbf{E}_i(p) = (p, \beta^{-1}(e_i))$.

A path $\alpha : I \to V$ is called an **integral curve** for a vector field $\mathbf{X}$ if $\mathbf{X} \circ \alpha = \dot{\alpha}$ whenever both sides are defined. In theory such curves can always be found:

Theorem 3. *Existence and uniqueness of integral curves (p.8)*
Let $\mathbf{X}$ be a smooth vector field on A and suppose $p \in A$. There exists an open interval $I \subset \mathbb{R}$ containing 0 and an integral curve $\alpha : I \to A$ for $\mathbf{X}$ such that

i. $\alpha(0) = p$

ii. If $\beta : \tilde{I} \to A$ is another integral curve with the same property then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

The directional derivative (p.54) $\nabla_v \mathbf{X}$ of a vector field $\mathbf{X}$ is the vector field with associated function $\nabla_v X$ where X is associated to $\mathbf{X}$. The **Lie bracket** $[\mathbf{X}, \mathbf{Y}]$ (p.182) of two vector fields $\mathbf{X}, \mathbf{Y}$ is the vector field given by $[\mathbf{X}, \mathbf{Y}](q) = \nabla_{\mathbf{X}(q)} \mathbf{Y} - \nabla_{\mathbf{Y}(q)} \mathbf{X}$.

A vector field is a choice of a vector V_p at each point. It is very convenient to also use the dual notion: a choice of a dual vector $(V_p)^*$ at each point. This is called a 1-form. Recall that the dual space is defined as $W^* = \{g : W \to \mathbb{R} | g \text{ is linear}\}$.

An important example of 1-forms is the differential. Given any function $f : A \to \mathbb{R}$ we may view df as a 1-form. Indeed, for every vector $v \in V^p$ we have $df(v) \in \mathbb{R}_{f(p)}$. It is customary to identify $\mathbb{R}_{f(p)}$ with $\mathbb{R}$ using the second coordinate. This makes df into a 1-form (see also the remark on p.109.).

More generally, a **differential k-form** (p.147) ω on A assigns a real number $\omega(v)$ to each $v \in (V_p)^k$, for $p \in A$ with the following properties. For any fixed $p \in A$ the map ω is multi-linear and alternating (skew-symmetric) in v. Multilinearity means that the function $v_i \mapsto \omega(v_1, \ldots, v_i, \ldots, v_k)$ is linear. Skewsymmetry means that ω changes sign when two arguments are interchanged. A k-form is said to be smooth if for all smooth vector fields $\mathbf{X_1}, \ldots \mathbf{X}_k$ the function $p \mapsto \omega(\mathbf{X}_1(p), \ldots \mathbf{X}_k(p))$ is smooth.

Differential k-forms (k-forms for short) arise naturally in differentiation and integration. In the former they appear when considering taking multiple (say

k) directional derivatives of a function. In integration they turn up as volume elements. Notice that volume of a block (parallelepiped) spanned by k vectors is computed by taking the determinant of the matrix whose columns are those vectors. The determinant is multi-linear and skew-symmetric in its columns.

A volume n-form on $A \subset V$ is an n-form that assigns ± 1 to each (ordered) orthonormal basis of V_p.

Given a basis $\beta : V \to \mathbb{R}^n$ we get coordinate functions $x^i = \pi^i \circ \beta : V \to \mathbb{R}$ where π^i means projection on the i-th coordinate of $\mathbb{R}^n$. It is convenient to work with the 1-forms dx^i. They are dual to the coordinate vector fields $\mathbf{E}_i$ in the sense that $dx^i(\mathbf{E}_j(p)) = \delta^i_j$ (Kronecker delta) for each i, j.

The **exterior derivative** of a 1-form ω on A is defined to be the 2-form $d\omega$ defined as follows. For $p \in A$ and $v_1, v_2 \in A_p$ set $d\omega(v_1, v_2) = \nabla_{v_1}\omega(V_2) - \nabla_{v_2}\omega(V_1) - \omega([V_1, V_2](p))$ where V_i is a smooth vector field with $V_i(p) = v_i$ defined on an open neighborhood of p.

This is definition is as beautiful as it is difficult to work with. Sometimes it is easier to just write the 1-form ω as $\omega = \sum_i f_i dx^i$ for some functions $f_i : A \to \mathbb{R}$ and use the facts that $ddx^i = 0$ and $d(f\eta) = df \wedge \eta + f d\eta$ and $d(\eta + \zeta) = d\eta + d\zeta$. This gives $d\omega = \sum_i df_i \wedge dx^i$.

Finally a decisive reason for working with differential forms instead of just vector fields is that differential forms are easier to transfer between spaces (change of coordinates). This is known as pull-back and does not work well for vector fields.

Given a map $\phi : A \to B$ the **pull-back** of a k-form ω on B is the k-form on A called $\phi^*\omega$ defined by $(\phi^*\omega)(v) = \omega(d\phi(v))$. Perhaps a better name would be change of coordinates.

2.2 Integration

We take for granted that for all open sets A there exists a map $\int_A :$ satisfying the following properties:

$\int_A$

Theorem 4. *Change of variables*
For a diffeomorphism $\phi : U \to \phi(U)$ we have

$$\int_U f \circ \phi |\det d\phi| = \int_{\phi U} f$$

3 Surfaces

The main object of study in this course are $n-k$-dimensional **surfaces** $S \subset V$ (p.126). By definition $S = f^{-1}(c)$ for some smooth $f : A \to \mathbb{R}^k$ such that df_p has rank k for every $p \in S$. The case $k = 1$ is central to the book. For the more delicate issues we restrict to $n = 3$ as this is where much of our intuition comes from.

A **parametrized** m-surface in V is a smooth map $\phi : U \to V$ where $U \subset W$ is a connected open subset of our m-dimensional space W. The purpose of parametrized surfaces is to transfer properties from open sets W onto surfaces. The key theorem is the following:

Theorem 5. *Parametrization of surfaces (p.121)*
For any $(n-k)$-surface $S \subset V$ and any $p \in S$ there exists an open $A \subset V$ and a parametrized $(n-k)$-surface $\phi : U \to V$ such that ϕ is an injection from U onto $A \cap S$.

4 Section analysis on surfaces

5 Riemannian geometry of surfaces