Exercises Manifolds 1

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Exercise 1 Polya vector field.
To make contact with complex analysis we use the bijection $\phi : \mathbb{R}^2 \to \mathbb{C}$ defined by $\phi(x, y) = x + iy$. Given a smooth function $f : V \to \mathbb{C}$ defined on some $V \subset \mathbb{C}$, George Polya cooked up the following vector field: X_f on $U = \phi^{-1}(V)$. For $p \in U$ he sets $X_f(p) = (p, \phi^{-1}(\bar{f}(\phi(p)))) \in \mathbb{R}^2_p$.
Show that $\text{rot}X_f = 0$ and $\text{div}X_f = 0$ if and only if f is holomorphic (complex differentiable).
Recall that the rotation rot for any vector field $Y(p) = (p, Y_1, Y_2)$ is defined as $\text{rot}Y = \partial_x Y_2 - \partial_y Y_1$.

Exercise 2 Find an explicit equation for a compact 2-surface in $\mathbb{R}^3$ with three holes. Recall compact in $\mathbb{R}^3$ means closed and bounded. You don't have to prove the number of holes is correct but you do have to show that the equation you give really produces a 2-surface.

Exercise 3 Find a vector field with a single zero such that going around the unit circle once the vector field makes two full rotations. (No proof required, just the equation).

Exercise 4 Describe the tangent space to the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$. When $n = 1$ how does it relate to the usual tangent line from calculus?

Exercise 5 Build a smooth function on $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x) = 1$ when $|x| < 1$ and $f(x) = 0$ when $|x| > 2$. You do need to prove that your f is smooth.

Exercise 6 Show that $\exp(At)$ is an integral curve for the system of linear ODE $\dot{x} = Ax$.

Exercise 7 Show that the Gauss map of a connected oriented n-surface that is the graph of a function Graph(f) (see Example 3 in Hoofdstuk 4) is NOT surjective.

Exercise 8 (Mobius strip). We may describe a Mobius strip in $\mathbb{R}^3$ as the image of the function $f : (-1, 1) \times [0, 2\pi] \to \mathbb{R}^3$ given by $f(s, t) = 2(\cos t, 0, \sin t) + (\cos(\frac{t}{2})(\cos(t), 0, \sin t) + \sin(\frac{t}{2})(0, 1, 0))s$. A normal at point $f(s_0, t_0)$ is given by $\mathbf{M}(s_0, t_0) = (f(s_0, t_0), M(s_0, t_0))$ where $M(s_0, t_0) = \partial_s f(s_0, t_0) \times \partial_t f(s_0, t_0)$. Show that M is continuous but $M(0, 0) \neq M(0, 2\pi)$. Explain why this means that the Mobius strip is NOT a 2-surface in $\mathbb{R}^3$.

Exercise 9 (Cone.) For $a > 0$ imagine a cone

$$C = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = a^2x^2 \text{and } x > 0\}$$

a. Write down the geodesic equation on C explicitly.

To better understand the cone we unwrap it as follows. Define the unwrapped cone as $U_\alpha = \{p \in \mathbb{R}^2 : \arg(p) \in [0, \alpha), |p| \neq 0\}$ here arg means the angle in $[0, 2\pi)$ of vector p with respect to the positive x-axis. The map $W_\alpha : U_\alpha \to \mathbb{R}^3$ that does the wrapping is

$$W_\alpha(p) = |p|(\sqrt{1 - \left(\frac{\alpha}{2\pi}\right)^2}, \frac{\alpha}{2\pi} \cos\left(\arg(p)\frac{2\pi}{\alpha}\right), \frac{\alpha}{2\pi} \sin\left(\arg(p)\frac{2\pi}{\alpha}\right))$$

b. Compute the correct value of $\alpha \in [0, 2\pi]$ such that W_α actually maps into and onto C. In other words W_α is a bijection onto C.

c. Verify that W_α sends straight line segments in U_α to geodesics in C.

Bonus. For what values of α can you make a geodesic intersect itself?

d. For $q = (1, a, 0)$, check that $v = (q, q) \in C_q$, the tangent space of C at point q.

e. Given the smooth curve $\gamma : [0, 2\pi] \to C$ defined by $\gamma(t) = (1, a\cos(t), a\sin(t))$, Compute the parallel transport $P_\gamma(v)$.

Exercise 10. (1-surfaces) Imagine a 1-surface S.

a. When is a curve $\alpha : I \to S$ a geodesic on S?

b. Explain why for any curve $\alpha : [a, b] \to S$ with $|\dot{\alpha}(t)| = 666$ the parallel transport is the unique linear map $P_\alpha : S_{\alpha(a)} \to S_{\alpha(b)}$ sending $\dot{\alpha}(a)$ to $\dot{\alpha}(b)$.

c. Recall the length of a curve $\alpha : [a, b] \to S$ is the integral $\int_a^b |\dot{\alpha}(t)|dt$. Now assume $S = \{(x, y) \in \mathbb{R}^2 | y = gx^2\}$ is a parabola for some $g \in \mathbb{R}$ and consider a curve $\alpha : [0, b] \to S$ with $|\dot{\alpha}(t)| = 1$. Give S an orientation and compute the length of the image of α under the corresponding Gauss map.

Exercise 11. Consider a spherical triangle T on $\mathbb{S}^2 \subset \mathbb{R}^3$ whose vertices are the north pole and two distinct points on the equator. Show that parallel transport around the three edges of T is the map that turns all vectors in the tangent space at the north pole by an angle ϕ. How is ϕ related to the area of T?

Exercise 12.

a. Write down the Weingarten map for the cylinder $C = \{p \in \mathbb{R}^3 : y^2 + z^2 = 1\}$ at the point $p_\theta = (0, \cos\theta, \sin\theta)$ for any fixed $\theta \in \mathbb{R}$.

b. What are the principal curvature directions and what are the corresponding principal curvatures at point p_θ? What is the Gaussian curvature?

c. Repeat parts a,b for the hyperboloid $H = \{p \in \mathbb{R}^3 : y^2 + z^2 = 1 + x^2\}$ at the same point p_θ.

d. Imagine a 2-surface $S \subset \mathbb{R}^3$ with the property that for each $p \in S$ there is a straight line contained in S passing through p. Prove that S has Gaussian curvature 0 or give a counterexample.

Exercise 13. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ as usual and defining $dz = dx + idy$ relate the 1-form η from Example 2 on p.75 of Thorpe to $\frac{dz}{z}$. Can you now explain Theorem 3 on p.76 using your knowledge of complex integration?

Exercise 14. Recall the chain rule as used in the book (e.g. p. 13): for every smooth curve $\alpha : I \to U \subset \mathbb{R}^n$ and a smooth function $f : U \to \mathbb{R}$ defined on open U we have $(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t)$. Using this prove the following more general version of the chain rule:
Given smooth maps $\phi : U \to V$ and $\psi : V \to W$ between open subsets $U \subset \mathbb{R}^k, V \subset \mathbb{R}^l, W \subset \mathbb{R}^m$ prove that the chain rule holds in the following form: $d(\psi \circ \phi) = d\psi \circ d\phi$.

Exercise 15. Show that the circle S^1 is a manifold by giving enough local parameterizations (or charts) to cover it. Bonus: Will one chart suffice? See the file additional material on the website for the definition of manifold.

Exercise 16. (When life is linear)

a. For a linear map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ and $(p, v) \in \mathbb{R}^n_p$ prove that $d\phi(p, v) = (\phi(p), \phi(v))$.

b. Prove the following special case of the change of variables theorem in two variables: For an invertible linear map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ linear and $U = (0,1)^2$, $\int_{\phi(U)} f = \int_U f \circ \phi |\det d\phi|$.

c. Compute $V(\psi)$ for the singular 3-surface $\psi : (0,1)^3 \to \mathbb{R}^4$ given by $\psi(x, y, z) = (2x - y, y - z, z - x, x + y)$.

d. Bonus (hard!): Can you compute the volume of a regular tetrahedron without integration?

Exercise 17. (Polyhedral Gauss-Bonnet)
Given a finite set of closed Euclidean triangles $T_1, \ldots, T_k \subset \mathbb{R}^3$ the union $S = \cup_i T_i$ is called a polyhedral surface if the following three conditions hold. No two triangles intersect in more than one of their sides. Every side of T_i belongs

to exactly one other triangle $T_j \neq T_i$. And finally, if v is the common vertex for the triangles $T_{i_1}, T_{i_2}, \dots T_{i_n}$ and s_{i_k} is the side opposite to v in triangle T_{i_k} then the union of the s_{i_k} is connected.

For each vertex v of S we define the (discrete) curvature $K(v)$ to be 2π minus the sum of angles directly adjacent to the vertex v.

a. Show that if S is a regular tetrahedron then $K(v) = \pi$ for each vertex v.

b. Prove that $\sum_{v \in V} K(v) = 2\pi\chi(S)$ where χ is the Euler characteristic $V - E + F = \chi(S)$. The number of edges is E and the number of faces (triangles) is F. This is a polyhedral version of the Gauss-Bonnet theorem where instead of integrating the curvature we concentrate it at the vertices and sum.

Exercise 18. (Surfaces with boundary)
Prove that the boundary of a 33-surface with boundary is a 32-surface. Does it have boundary? Is every n-surface without boundary, the boundary for some $n+1$ surface with boundary?

Exercise 19. (Induced orientation)
Choose a volume form ζ on the disk $D(r) = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 \leq r^2\}$ and describe what the induced orientation 1-form looks like in the point $(r, 0, 0)$.

Exercise 20. (Partition of unity)
Choose two local parameterizations ϕ_1, ϕ_2 of the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ whose images cover $\mathbb{S}^1$. Describe explicitly the partition of unity subordinate to ϕ_1, ϕ_2 by giving formulas for the functions f_1, f_2.

Exercise 21. (Forms in local coordinates)
Suppose $B_1, .., B_n$ are smooth tangent vector fields on n-surface $S \subset \mathbb{R}^{n+1}$ such that for each $p \in S$ the tangent vectors $B_1(p), .., B_n(p) \in S_p$ are linearly independent.

a. Prove that for each $i = 1, .., n$ there exists a unique 1-form β_i such that $\beta_i(B_j) = \delta_{i,j}$ (Kronecker delta).

b. Show that the (smooth) k-forms on S form an $\mathbb{R}$-vector space $\Omega_k(S)$.

c. Prove that for each $\omega \in \Omega_2(S)$ there exist unique smooth functions $c_{i,j} : S \to \mathbb{R}$ such that $\omega = \sum_{i<j} c_{i,j} \beta_i \wedge \beta_j$.

d. For $\omega \in \Omega_1(S)$ explain how the exterior derivative $d\omega$ is expressed in terms of derivatives of the coefficients of ω with respect to the β_i.

e. What is the dimension of $\Omega_n(S)$?

Exercise 22. (The Key lemma for the cylinder)
Verify Lemma 3 on p.193 of Thorpe explicitly for the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : (x+z)^2 + (y+z)^2 = 1\}$ by computing both the left hand side and the right

hand side explicitly.

Exercise 23. (Not exactly exact)
Find a surface and a smooth 2-form on it that is NOT of the form $d\omega$ for any smooth 1-form on the surface.
Exercise 24. (Smooth connection form)
Prove that the connection 1-form is smooth (see chapter 21, p.190).
Exercise 25. (Torus)
Construct a smooth tangent unit vector field X on the torus T from example 8, p.112. Work out precisely what the corresponding connection 1-form ω does to the coordinate vector fields. Now find a parallel tangent vector field Z, non-zero and distinct from X. How much does Z rotate with respect to X when moving along the curve $\alpha : [0, 2pi] \to T$ given by $\alpha(t) = \varphi(2t, 3t)$? What does this have to do with our 1-form ω?