Differentiable Manifolds Homework Solutions

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1 Homework 1 (in dutch)

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Opgave 2c

Zij $L : \mathbb{R}^n \to \mathbb{R}^n$ een lineaire afbeelding. We gaan bewijzen dat L continu is.

Lemma: Zij $f : \mathbb{R}^n \to \mathbb{R}^n$ een afbeelding en $a \in \mathbb{R}^n$. Stel, voor elke $\epsilon > 0$ is er een $\delta > 0$ zodat voor alle $x \in \mathbb{R}^n$ met $||x - a|| < \delta$ geldt $||f(x) - f(a)|| < \epsilon$. Dan is f continu in de zin dat voor alle open $A \subset \mathbb{R}^n$ ook $f^{-1}(A)$ open is.

Bewijs: Dit volgt uit Theorem 2.3.7 en Corollary 2.3.10 van A Taste of Topology van Volker Runde. □

Omdat L lineair is, is er een $k > 0$ zodat voor alle $x \in \mathbb{R}^n$ geldt dat $||L(x)|| \leq k||x||$, zoals bekend is van Lineaire Algebra. Zij $a \in \mathbb{R}^n$ en zij $\epsilon > 0$. Neem nu $\delta := \frac{1}{k}\epsilon$. Voor alle $x \in \mathbb{R}^n$ met $||x - a|| < \delta$ geldt nu

$$\begin{aligned} ||L(x) - L(a)|| &= ||L(x - a)||, \text{ want } L \text{ is lineair,} \\ &\leq k||x - a|| < k \cdot \frac{1}{k}\epsilon = \epsilon. \end{aligned}$$

Met lemma A volgt dat L continu is in a. Omdat a arbitrair was, concluderen we dat L continu is.

Opgave 3b

Zij $F \in L(U, V)$ en $G \in L(V, W)$. We gaan aantonen dat $(G \circ F)^* = F^* \circ G^*$. Zij $\phi \in W^*$ willekeurig gegeven. Dan geldt

$$\begin{aligned} (G \circ F)^*(\phi) &= \phi \circ (G \circ F), \text{ per definitie van } (G \circ F)^*, \\ &= \phi \circ G \circ F, \text{ want samenstelling van afbeeldingen is associatief.} \end{aligned}$$

Verder geldt

$$\begin{aligned} (F^* \circ G^*)(\phi) &= F^*(G^*(\phi)) \\ &= F^*(\phi \circ G), \text{ per definitie van } G^*, \\ &= (\phi \circ G) \circ F, \text{ per definitie van } F^*, \\ &= \phi \circ G \circ F, \text{ want samenstelling van afbeeldingen is associatief.} \end{aligned}$$

We zien dat $(F^*\circ G^*)(\phi) = \phi\circ G\circ F = (G\circ F)^*(\phi)$. Omdat ϕ arbitrair was, volgt $F^*\circ G^* = (G\circ F)^*$.

Opgave 4c

We noteren $E_2 := (e_1, e_2)$ en $E_3 := (e_1, e_2, e_3)$ voor de standaardbases van $\mathbb{R}^2$ resp. $\mathbb{R}^3$. Noteer $B := (e_1 + e_2, e_1 - e_2)$ voor de andere gegeven basis van $\mathbb{R}^3$. Noteer $[DF(0,\pi)]_{E_3}^{E_2}$ voor $DF(0,\pi)$ t.o.v. de standaardbases. Volgens het dictaat van dit vak is $[DF(0,\pi)]_{E_3}^{E_2}$ de Jacobiaan van F in $(0,\pi)$ (onder de definitie van partiele afgeleides staat $D_j f^i = \frac{\partial f^i}{x_j}$, en dat $D_j f^i$ op plek (i,j) van $DF(a)$ staat.) We hebben dus

$$\begin{aligned}[DF(0,\pi)]_{E_3}^{E_2} &= \begin{pmatrix} \frac{\partial F^1}{x}(0,\pi) & \frac{\partial F^1}{y}(0,\pi) \\ \frac{\partial F^2}{x}(0,\pi) & \frac{\partial F^2}{y}(0,\pi) \\ \frac{\partial F^3}{x}(0,\pi) & \frac{\partial F^3}{y}(0,\pi) \end{pmatrix} \\ &= \begin{pmatrix} 2x+2y & 2x \\ 0 & -\sin(y) \\ 1 & 0 \end{pmatrix} |_{(x,y)=(0,\pi)} = \begin{pmatrix} 2\pi & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Door een basistransformatie toe te passen berekenen we nu de gevraagde matrix $[DF(0,\pi)]_{E_3}^{B}$ van $DF(0,\pi)$ t.o.v. de bases B en E_3. De basistransformatiematrix van E_2 naar B_2 is $[id_{\mathbb{R}^2}]_{E_2}^{B}$, in de zin dat $[DF(0,\pi]_{E_2}^{B} = [DF(0,\pi)]_{E_3}^{E_2} \cdot [id_{\mathbb{R}^2}]_{E_2}^{B}$. We hebben

$$[id_{\mathbb{R}^2}]_{E_2}^{B} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Er volgt

$$\begin{aligned}[DF(0,\pi]_{E_2}^{B} &= [DF(0,\pi)]_{E_3}^{E_2} \cdot [id_{\mathbb{R}^2}]_{E_2}^{B} \\ &= \begin{pmatrix} 2\pi & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2\pi & 2\pi \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.\end{aligned}$$

Dus de matrix van $DF(0,\pi)$ t.o.v. bases B en E_3 is $\begin{pmatrix} 2\pi & 2\pi \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Opgave 5b

We willen een C^2-functie $f : \mathbb{R} \to [0,1]$ contstrueren zodanig dat $f|_{[-1,1]} = 1$ en $f(x) = 0$ voor $|x| > 2$. Aangezien f op $(-\infty, 2), [-1,1]$ en op $(2,\infty)$ al vastligt, moeten we f alleen nog definiëren op $[-2,-1)$ en $(1,2]$. We willen f definiëren op $[-2,-1)$ en $(1,2]$ door $f(x) = g(x)$ als $-2 \le x < -1$ en $f(x) = h(x)$ als $1 < x \le 2$. Om f een C^2-functie naar $[0,1]$ te laten zien, moeten h en g voldoen aan $g([-2,-1) = [0,1] = h(1,2]$ en moeten g en h ook C^2 zijn op $[-2,-1)$ resp. $(1,2]$. Verder moeten g en h voldoen aan

$$g(-2) = 0, g(-1) = 1, g^{(i)}(x) = 0 \text{ voor } i = 1,2 \text{ en } x = -2,-1,$$

en

$$h(1) = 1, h(2) = 0, h^{(i)}(x) = 0 \text{ voor } i = 1,2 \text{ en } x = 1,2,$$

want in dat geval zijn f, f' en f'' ook continu in -2,-1,1 en 2. Als we een geschikte g hebben, voldoet $h(x) = g(-x)$ ook aan de eisen wegens symmetrie in de y-as.

Beschouw de functie
$$y(x) = x - \frac{\sin(2\pi x)}{2\pi}.$$
Hiervoor geldt
$$y'(x) = 1 - \cos(2\pi x) \text{ en } y''(x) = 2\pi \sin(2\pi x).$$
Merk op dat $y(0) = 0$, $y(1) = 1$ en $y^{(i)}(x) = 0$ voor $i = 1, 2$ en $x = 0, 1$. Ook geldt voor alle $x \in [0, 1]$ dat $\cos(2\pi x) \leq 1$, dus $y'(x) \leq 0$. Dus op $[0, 1]$ stijgt y van 0 naar 1. Door y nu 2 naar links te transleren, krijgen we onze g met de gewenste eigenschappen:
$$g(x) = y(x + 2) = x + 2 - \frac{\sin(2\pi(x + 2))}{2\pi}.$$
Zoals eerder opgemerkt kunnen we voor h nu nemen:
$$h(x) = g(-x) = -x + 2 - \frac{\sin(2\pi(-x + 2))}{2\pi}.$$
We concluderen dat we f als volgt kunnen definiëren:
$$f(x) = \begin{cases} 0 & \text{,als } x < -2, \\ x + 2 - \frac{\sin(2\pi(x+2))}{2\pi} & \text{,als } -2 \leq x < -1, \\ 1 & \text{,als } -1 \leq x \leq 1, \\ -x + 2 - \frac{\sin(2\pi(-x+2))}{2\pi} & \text{,als } 1 < x \leq 2, \\ 0 & \text{,als } x > 2. \end{cases}$$
Deze $f : \mathbb{R} \to [0, 1]$ is C^2 en voldoet aan $f|_{[-1,1]} = 1$ en $f(x) = 0$ voor $|x| > 2$.

2 Homework 2 (in dutch)

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Opgave 2.1.4

Vraag a

Zij $X \subset V$ en $Y \subset W$ manifolds, zeg van dimensies n resp. m. We gaan laten zien dat $X \times Y \subset V \times W$ ook een manifold is, van dimensie $m + n$. Merk hiertoe op: per definitie is een deelverzameling $A \subset Z$ van een vectorruimte Z een k-dimensionaal manifold als A een vereniging is van k-dimensionale coordinate patches. Equivalent hieraan is dat iedere $a \in A$ een element is van een k-dimensionale coordinate patch in X. Deze tweede karakterisering zullen we nu gebruiken.

Zij $(x, y) \in X \times Y$. Kies een n-dimensionale coordinate patch C_1 van X met $x \in C_1$. Kies ook een m-dimensionale coordinate patch C_2 van Y met $y \in C_2$. Dit geeft $(x, y) \in C_1 \times C_2 \subset X \times Y$. We gaan aantonen dat $C_1 \times C_2$ een $n + m$-dimensonale coordinate patch in $X \times Y$ is. Laat $\psi_1 : B_1 \to C_1$ en $\psi_2 : B_2 \to C_2$ de bij de coordinate patches horende diffeomorfismes zijn, met $B_1 \subset F$ en $B_2 \subset G$ open deelverzamelingen van vectorruimtes F en G van dimensie n resp. m.

Laat $0 \leq k, l \leq \infty$ maximaal zijn zodat ψ_1 en ψ_2 resp. c^k- en c^l-diffeomorfismes zijn. Laat $p := \min\{k, l\}$.

Nu is $B_1 \times B_2$ op zijn beurt weer open in de $(n+m)$-dimensionale vectorruimte $F \times G$, per definitie van de producttopologie. Zo is ook $C_1 \times C_2$ open in $V \times W$, omdat C_1 en C_2 als zijnde coordinate patches open zijn in C_1 resp. C_2. Definiëer nu de afbeelding

$$\psi : B_1 \times B_2 \to C_1 \times C_2, \text{ gegeven door } \psi(b_1, b_2) = (\psi_1(b_1), \psi_2(b_2)).$$

Merk op dat ψ de volgende inverse heeft:

$$\psi^{-1} : C_1 \times C_2 \to B_1 \times B_2, \text{ gegeven door } \psi^{-1}(c_1, c_2) = (\psi_1^{-1}(c_1), \psi_2^{-1}(c_2)).$$

Merk op dat $\psi_1, \psi_2, \psi_1^{-1}$ en ψ_2^{-1} alle c^p-afbeeldingen zijn. Omdat we ψ en ψ^{-1} coördinaatsgewijs kunnen differentiëren, volgt dat ψ en ψ^{-1} beide c^p-afbeeldingen zijn. Dus psi is een c^p diffeomorfisme. Hiermee os bewezen dat $C_1 \times C_2$ een $(n+m)$-dimensionale coordinate patch is van $X \times Y$, met $(x, y) \in C_1 \times C_2$.

Deze redenering laat zien dat $X \times Y$ een $(n+m)$-dimensionaal manifold is.

Vraag c

In deze opgave laten we zien dat $X = \{(x, y, z, w) \in \mathbb{R}^4 : x^4 + y^2 - w^2 = 1, e^z + e^w = 1\}$ een manifold is, d.m.v. de Rank Theorem.

Beschouw de functie

$$f : \mathbb{R}^4 \to \mathbb{R}^2, \ f(x, y, z, w) = (x^4 + y^2 - w^2, e^z + e^w).$$

Merk op dat $X = f^{-1}(1, 1)$. Merk op dat f een c^∞-afbeelding is. We hebben

$$Df(x, y, z, w) = \begin{pmatrix} 4x^3 & 2y & 0 & -2w \\ 0 & 0 & e^w & e^z \end{pmatrix}.$$

We hebben $Df(x, y, z, w)e_3 = (0, e^w)$, dus $Df(x, y, z, w)$ spant in ieder geval de tweede coördinaatas op. Verder hebben we

$$Df(x, y, z, w)e_1 = (4x^3, 0), \ Df(x, y, z, w)e_2 = (2y, 0) \text{ en } Df(x, y, z, w)e_2 = (-2w, 0).$$

Dus als 1 van x, y en w niet 0 is, dan spant $Df(x, y, z, w)$ ook de eerste coördinaatas op en dan geldt dus $\mathrm{Rk}(Df(x, y, z, w)) = 2$. Laat nu $Y := \{(x, y, z, w) \in \mathbb{R}^4 : x \neq 0 \text{ of } y \neq 0 \text{ of } w \neq 0\}$. Merk op dat Y open is in $\mathbb{R}^4$. Laat

$$g : Y \to \mathbb{R}^2, \ g(x, y, z, w) = f(x, y, z, w),$$

De beperking van f tot Y zijn. Ook g is c^∞ en op Y heeft $Dg(x, y, z, w)$ constante rank 2. Met de inverse functiestelling volgt dat $g^{-1}(1, 1)$, mits niet leeg, een c^∞-manifold is van dimensie $\dim(\mathbb{R}^4) - 2 = 2$. We hebben

$$\begin{aligned} g^{-1}(1,1) &= \{(x, y, z, w) \in Y : g(x, y, z, w) = (1, 1)\} \\ &= \{(x, y, z, w) \in \mathbb{R}^4 : x \neq 0 \text{ of } y \neq 0 \text{ of } w \neq 0 \text{ en } f(x, y, z, w) = (1, 1)\} \\ &= \{(x, y, z, w) \in \mathbb{R}^4 : x \neq 0 \text{ of } y \neq 0 \text{ of } w \neq 0 \text{ en } x^4 + y^2 - w^2 = 1, e^z + e^w = 1\} \\ &= \{(x, y, z, w) \in \mathbb{R}^4 : x^4 + y^2 - w^2 = 1, e^z + e^w = 1\} = X, \end{aligned}$$

want de vergelijking $x^4 + y^2 - w^2 = 1$ impliceert al dat $x \neq 0$ of $y \neq 0$ of $w \neq 0$. Dus $X = g^{-1}(1, 1)$ is een 2-dimensionaal manifold.

Opgave 2.2.1

Vraag a

Merk op dat de samenstelling $\phi^{-1} \circ \phi'$ niet bestaat: ϕ^{-1} is gedefiniëerd op C, terwijl $\phi'(B') = C'$. Echter, $C \cap C' \neq \emptyset$, want $x \in C \cap C'$. Door ϕ' te beperken tot $\phi'^{-1}(C)$, krijgen we een diffeomorfisme $\phi'|_{\phi'^{-1}(C)}$ van $\phi'^{-1}(C)$ naar $C' \cap C$, omdat phi' bijectief van B' naar C' gaat. Dit kunnen we vervolgens wel samenstellen met ϕ^{-1}, want $C' \cap C \subset C$. Deze samenstelling is een diffeomorfisme $F = \phi^{-1} \circ \phi'|_{\phi'^{-1}(C)}$, omdat F een samenstelling is van diffeomorfismes. Om consistent te blijven met de notatie van de opgave zullen we F wel noteren als $F = \phi^{-1} \circ \phi'$. Het schrijven van $F = \phi^{-1} \circ \phi'|_{\phi'^{-1}(C)}$ verandert namelijk niets aan de verdere bewijzen. Omdat F een diffeomorfisme is, volgt met Lemma 1.2.3 (Derivative of Diffeomorphism) direct dat $DF(b') : W \to W$ een lineair isomorfisme is.

Vraag b

Per definitie hebben we $F = \phi^{-1} \circ \phi'$. Beide kanten van links met ϕ samenstellen geeft $\phi \circ F = \phi'$. Aan beide kanten de afgeleide nemen geeft nu

$$D\phi'(b') = D(\phi \circ F)(b'). \tag{1}$$

De kettingregel, lemma 1.2.1 van het dictaat, geeft

$$\begin{aligned} D(\phi \circ F)(b') &= D\phi(F(b')) \circ DF(b') \\ &= D\phi(b) \circ DF(b'), \end{aligned}$$

want $F(b') = \phi^{-1}(\phi'(b')) = \phi^{-1}(x) = b$. Invullen in (1) geeft de gewenste vergelijking

$$D\phi'(b') = D\phi(b) \circ DF(b').$$

Vraag c

In de vergelijking $D\phi'(b') = D\phi(b) \circ DF(b')$, die we in vraag b bewezen, nemen we aan beide kanten het beeld:

$$\mathrm{Im}\left(D\phi'(b')\right) = \mathrm{Im}\left(D\phi(b) \circ DF(b')\right). \tag{2}$$

In vraag a merkten we al op dat $DF(b')$ een isomorfisme is. Dit geeft

$$\mathrm{Im}\left(D\phi(b) \circ DF(b')\right) = \mathrm{Im}\left(D\phi(b)\right).$$

Dit invullen in 2 geeft inderdaad $\mathrm{Im}\left(D\phi'(b')\right) = \mathrm{Im}\left(D\phi(b)\right)$. Via ϕ zouden we T_xX definiëren als $T_xX = \mathrm{Im}(D\phi(b))$ en via ϕ' zouden we T_xX definiëren als $T_xX = \mathrm{Im}(D\phi'(b'))$. De gelijkheid $\mathrm{Im}\left(D\phi'(b')\right) = \mathrm{Im}\left(D\phi(b)\right)$ laat zien dat deze definitie niet afhangt van de keuze van de coordinate patch.

Opgave 2.2.2.a

Zij $X \subset V$ een open deelverzameling van een vectorruimte V met $x \in X$. Dan is X een manifold met 1 coordinate patch, namelijk X zelf, met parametrisatie

$$id_X : X \to X, a \mapsto a.$$

Immers, X is open in X, X is een open deelverzameling van de vectorruimte V, en id_X is c^∞ met c^∞-inverse $id_X^{-1} = id_X$, dus id_X is een c^∞-diffeomorfisme. Voor alle $a \in X$ hebben we $Did_X(a) = id_V \in L(V, V)$. Immers, voor alle $a \in X$ geldt

$$\lim_{v\to 0} \frac{|id_X(a+v) - id_X(a) - id_X(v)|}{v} = \lim_{v\to 0} \frac{|a+v-a-v|}{|v|} = 0.$$

Omdat $id_X(x) = x$, hebben we

$$T_xX = \mathrm{Im}(Did_X(x)) = \mathrm{Im}(Did_X(x)) = \mathrm{Im}(id_V) = V.$$

Dus T_xX is de hele vectorruimte V.

3 Homework 3

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Exercise 6b
Let $M(t) = \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix}$ and let $f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$. Now we have the following

$$\begin{aligned} D_1Mf(t) = D_1 \begin{pmatrix} x(t)f_1(t) + y(t)f_2(t) \\ z(t)f_1(t) + w(t)f_2(t) \end{pmatrix} &= \begin{pmatrix} \frac{d}{dt}(x(t)f_1(t) + y(t)f_2(t)) \\ \frac{d}{dt}(z(t)f_1(t) + w(t)f_2(t)) \end{pmatrix} \\ &= \begin{pmatrix} x(t)\frac{d}{dt}f_1(t) + f_1(t)\frac{d}{dt}x(t) + y(t)\frac{d}{dt}f_2(t) + f_2(t)\frac{d}{dt}y(t) \\ z(t)\frac{d}{dt}f_1(t) + f_1(t)\frac{d}{dt}z(t) + w(t)\frac{d}{dt}f_2(t) + f_2(t)\frac{d}{dt}w(t) \end{pmatrix} \\ &= \begin{pmatrix} x(t)\frac{d}{dt}f_1(t) + y(t)\frac{d}{dt}f_2(t) \\ z(t)\frac{d}{dt}f_1(t) + w(t)\frac{d}{dt}f_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t)\frac{d}{dt}x(t) + f_2(t)\frac{d}{dt}y(t) \\ f_1(t)\frac{d}{dt}z(t) + f_2(t)\frac{d}{dt}w(t) \end{pmatrix} \\ &= \begin{pmatrix} x(t) & y(t) \\ z(t) & w(t) \end{pmatrix} \begin{pmatrix} \frac{d}{dt}f_1(t) \\ \frac{d}{dt}f_2(t) \end{pmatrix} + \begin{pmatrix} \frac{d}{dt}x(t) + \frac{d}{dt}y(t) \\ \frac{d}{dt}z(t) + \frac{d}{dt}w(t) \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = (D_1M(t))f(t) + M(t)(D_1f(t)). \end{aligned}$$

Exercise 3a
We can rewrite F as follows

$$F(\theta, t) = \begin{pmatrix} \cos(\theta)(1 + t\cos\frac{\theta}{2}) \\ \sin(\theta)(1 + t\cos\frac{\theta}{2}) \\ t\sin\frac{\theta}{2} \end{pmatrix}.$$

Here it is easy to see that the inverse functions of ϕ and ψ are now given by

$$\phi^{-1} : M \to (-\pi, \pi) \times (\frac{1}{2}, \frac{1}{2}), \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tan^{-1}(\frac{y}{x}) \\ \frac{z}{\sin(\frac{\tan^{-1}(\frac{y}{x})}{2})} \end{pmatrix}.$$

$$\psi^{-1} : M \to (0, 2\pi) \times (\frac{1}{2}, \frac{1}{2}), \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tan^{-1}(\frac{y}{x}) \\ \frac{z}{\sin(\frac{\tan^{-1}(\frac{y}{x})}{2})} \end{pmatrix}.$$

So we finally conclude that ϕ and ψ are diffeomorphism with inverse functions ϕ^{-1} and ψ^{-1}. Since we also realise that the domain of ϕ and ψ are both open in $\mathbb{R}^2$ we conclude that ϕ and ψ are both parametrizations of M whose image cover M.

Exercise 3b
By exercise $3a$ we know that M is a $2-$dimensional manifold, because it is a union of $2-$dimensional coordinate patches.

Exercise 3c
We have the following

$$DF = \begin{pmatrix} -\sin\theta - \frac{t}{2}\sin\frac{\theta}{2}\cos\theta - t\cos\frac{\theta}{2}\sin\theta & \cos\frac{\theta}{2}\cos\theta \\ \cos\theta + \frac{t}{4}\cos\frac{\theta}{2} + \frac{3t}{4}\cos\frac{3\theta}{2} & \cos\frac{\theta}{2}\sin\theta \\ \frac{t}{2}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$$

Thus it holds that

$$DF(\theta, 0) = \begin{pmatrix} -\sin\theta & \cos\frac{\theta}{2}\cos\theta \\ \cos\theta & \cos\frac{\theta}{2}\sin\theta \\ 0 & \sin\frac{\theta}{2}. \end{pmatrix}$$

Now let $B = (b_1, b_2)$ be the basis for the tangent space such that

$$b_1 = DF(\theta, 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

$$b_2 = DF(\theta, 0)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Now note that $0 = b_1 b_2 = x\sin^2\theta + x\cos^2\theta = x$.
So we know $x = 0$, thus $b_2 = DF(\theta, 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta \\ \cos\frac{\theta}{2}\sin\theta \\ \sin\frac{\theta}{2} \end{pmatrix}$.

Exercise 3e
We have the following for g

$$Dg(x, y, z) = \begin{pmatrix} yze^{xyz} & xze^{xyz} & xye^{xyz} \\ 1 & 0 & 0 \end{pmatrix}$$

Thus it follows that

$$Dg(1, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since we also know that $F(\theta, t) = (1, 0, 0)$ for $\theta = t = 0$ we choose basis $B = (\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})$.

So we know conclude that the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

is the linear transformation $Dg(1, 0, 0)$ with respect to basis B.

Exercise 1
For a geodesic $\gamma : [a, b] \to V$ we know that $\ddot{\gamma}(t) \perp T_{\gamma}(t)V$. By the previous homework we then know $\ddot{\gamma}(t) \perp V$. So $\ddot{\gamma}(t) = 0$ and thus we conclude that the geodesics of V are all the polynomials of degree less than 1 (linear and constant functions).

4 Homework 4

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2.4.1(a)
First we want to check wether G is wel defined. We know that for $b \in B$ that $D\phi(b) : W \to V$ and therefore $(D\phi(b))^{-1} : V \to W$. Since $\phi(b) \in C$ and since F is a tangent vector field , we know that $F(\phi(b)) \in T_{\phi(b)}X$, so $F(\phi(b)) \in V$. Thus we conclude that G is wel defined.

Now we want to look at the differentiability for G. We know already that ϕ is a diffeomorphism. Therefore $(D\phi(b))^{-1}$ is a differentiable matrix and $F(\phi(b))$ is a differentiable vector. Now we have that $G(b) = (D\phi(b))^{-1}F(\phi(b))$ is also differential by Exercise 1.2.6.

2.4.1(b)
Since $B \subset W$ is open, we know by Theorem 2.4.1 that we can find an integral curve $\alpha : [-\epsilon, \epsilon] \to B$ for G with G with $\alpha(0) = b$.

2.4.1(c)
By the defintion of an integral curve we want to show that $D\gamma(t)(e_1) = F(\gamma(t))$ for all $t \in [-\epsilon, \epsilon]$. Since α is integral curve for G we know the following for all $t \in [-\epsilon, \epsilon]$

$$\begin{aligned} D\gamma(t)(e_1) &= D(\phi \circ \alpha)(t)(e_1) \\ &= \big((D\phi)(\alpha(t)) \circ (D\alpha)(t)\big)(e_1) \\ &= (D\phi)(\alpha(t))(G(\alpha(t))) \\ &= (D\phi)(\alpha(t))\big((D\phi)(\alpha(t))\big)^{-1}F(\phi(\alpha(t))) \\ &= F(\phi(\alpha(t))) = F(\gamma(t)). \end{aligned}$$

Thus we conclude that γ is an integral curve for F. Now since there always exists an integral curve for G (ODE theorem) and a coordinate patch for the manifold, we proved by this question the existence part of our theorem.

2.4.2
Take a look at the formula $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \to (x^2 - y^2, 2xy)$. Note that it satisfies the conditions that $F = 0$ only at the point $(0, 0)$ and that for any $r > 0$ and t from 0 to 2π $\frac{F(r\cos t, r\sin t)}{|F(r\cos t, r\sin t)|}$ rotates around the unit circle twice in the counter clockwise direction.

2.5.1(a)
We know that $\phi^{-1} : X \to \mathbb{R}^2, (x, y, z) \mapsto (x, y)$, so clearly ϕ and ϕ^{-1} are bijective and C^∞. Now since we also know that the coordinate patches are 2−dimensional we conclude that X is a 2−manifold.

2.5.1(b)
We have the following

$$D\phi(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2\alpha y \end{pmatrix}.$$

Thus we can choose $b_1 = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}$ and $b_2 = \begin{pmatrix} 0 \\ 1 \\ 2\alpha y \end{pmatrix}$ as basisvectors for the tangent space $T_p X$ with $p = (x, y, z)$.

2.5.1(c)
For a unit normal vector field we must find an $n = (a, b, c)$ such that $nb_1 = 0$ and $nb_2 = 0$. This gives us the following system of equations

$$\begin{cases} a + 2cx = 0 \\ b + 2c\alpha y = 0 \end{cases}.$$

Note that $n = (a, b, c) = (2x, 2\alpha y, -1)$ satisfies this system and thus we find that the unit normal vector field is given by

$$\mathcal{N} : X \to S^2, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{1}{\sqrt{4x^2 + 4\alpha^2 y^2 + 1}} \begin{pmatrix} 2x \\ 2\alpha y \\ -1 \end{pmatrix}.$$

2.5.1(d)
We know that $T_p X$ is orthogonal to $\mathcal{N}(p)$. But we also see that $T_{\mathcal{N}(p)} S^2$ is everything that is orthogonal to $\mathcal{N}(p)$. Therefore the orthogonal complement of $\mathcal{N}(p)$ is equal to $T_p X = T_{\mathcal{N}(p)} S^2$ for any point p.

2.5.1(e)
We have the following for $M = \sqrt{4x^2 + 4\alpha^2 y^2 + 1}$

$$D\mathcal{N}(x, y, z) = \begin{pmatrix} \frac{2M - \frac{16x^2}{M}}{M^2} & \frac{-16\alpha^2 xy}{M^3} & 0 \\ \frac{-16\alpha xy}{M^3} & \frac{2\alpha M - \frac{16\alpha^3 y^2}{M}}{M^2} & 0 \\ \frac{8x}{M^3} & \frac{8\alpha^2 y}{M^3} & 0 \end{pmatrix}.$$

Now we want to find $D\mathcal{N}(x, y, z)$ with respect to basis (b_1, b_2) so we calculate

$$B^\top D\mathcal{N}(x, y, z) B = \begin{pmatrix} \frac{2}{M} & 0 \\ 0 & \frac{2\alpha}{M} \end{pmatrix}.$$

Here is B the 3×2 matrix with b_i as the $i-$th column for $i \in \{1, 2\}$.
Now we finally conclude that $D\mathcal{N}(0, 0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2\alpha \end{pmatrix}$.

2.5.1(f)
The curvature at $(0, 0, 0)$ of X is equal to the following

$$\mathcal{K}(0, 0, 0) = \det D\mathcal{N}(0, 0, 0) = 4\alpha.$$

5 homework 5

Author: Bart Eggen.

Exercise 2.6.1

Recall that we had the Möbius strip defined as the image of the function

$$F(\theta, t) = \begin{pmatrix} \cos\theta + t\cos\left(\frac{\theta}{2}\right)\cos\theta \\ \sin\theta + t\cos\left(\frac{\theta}{2}\right)\sin\theta \\ t\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

when $0 \leq \theta < 2\pi$ and $-\frac{1}{2} < t < \frac{1}{2}$. Then we calculated that a basis for the tangent space in the point $F(\theta, 0)$ is given by

$$\left(\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\cos\theta \\ \cos\left(\frac{\theta}{2}\right)\sin\theta \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \right).$$

We now want to prove that the Möbius strip is not orientable. To do this we first assume that it is orientable. So we know there exists a differential 2-form (as the dimension of M is 2) $\omega : X \to \mathrm{Alt}^2(\mathbb{R}^3)$ that is nowhere 0. We also know that this map is continuous. Let us now look at $\omega(F(\theta, 0)) : T_{F(\theta,0)}M \times T_{F(\theta,0)}M \to \mathbb{R}$. Especially look at the case that $\theta = 0$. We then have the basis $b_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of $T_{F(\theta,0)}M$. Because $\omega(F(0,0))$ is alternating and bilinear, we know that $\omega(F(0,0))(v, v) = 0$ for all $v \in T_{F(\theta,0)}M$ and $\omega(F(0,0))(v, w) = -\omega(F(0,0))(w, v)$ for all $v, w \in T_{F(\theta,0)}M$. So for two elements of $T_{F(\theta,0)}M$, denoted as $v = \lambda_1 b_1 + \lambda_2 b_2$ and $w = \mu_1 b_1 + \mu_2 b_2$ we have (leaving out some steps in between but it is just simple linearity and alternating properties)

$$\omega(F(0,0))(v, w) = \omega(F(0,0))(\lambda_1 b_1 + \lambda_2 b_2, \mu_1 b_1 + \mu_2 b_2) = (\lambda_1\mu_2 - \lambda_2\mu_1)\omega(F(0,0))(b_1, b_2))$$

Now we want to look what happens if we let $\theta \to 2\pi$. We know that $F(2\pi, 0) = F(0, 0)$ and we know that ω is a C^1 map so this means that it is continuous and we have

$$\lim_{\theta\to 2\pi} \omega(F(\theta, 0))(v, w) = \omega(F(2\pi, 0))(v, w) = \omega(F(0,0))(v, w).$$

In the limit we also know that the basis of the tangent space goes to $b_1, -b_2$ (just plug in 2π in the basis). So this means that

$$\begin{aligned} \omega(F(0,0))(v, w) &= \lim_{\theta\to 2\pi} \omega(F(\theta, 0))(v, w) \\ &= \lim_{\theta\to 2\pi} \omega(F(2\pi, 0))(v, w) = (\lambda_1\mu_2 - \lambda_2\mu_1)\omega(F(2\pi, 0))(b_1, -b_2)) \\ &= -(\lambda_1\mu_2 - \lambda_2\mu_1)\omega(F(2\pi, 0))(b_1, b_2)) = -(\lambda_1\mu_2 - \lambda_2\mu_1)\omega(F(0,0))(b_1, b_2)) \end{aligned}$$

So we see that $\omega(F(0,0))(v, w) = -\omega(F(0,0))(v, w)$, so $\omega(F(0,0))(v, w) = 0$ for all $v, w \in T_{F(\theta,0)}M$. So this means that $\omega(F(0,0))$ is 0 which is in contradiction with what we assumed. So we see that M is not orientable.

Exercise 2.6.2

Let V be n-dimensional the an element of $\mathrm{Alt}^n(V)$ is given by $f : V^n \to \mathbb{R}$ where we have the properties that $f(v_1, ..., v, ..., v, ..., v_n) = 0$ and $f(v_1, ..., v_i, ..., v_j, ..., v_n) = -f(v_1, ..., v_j, ..., v_i, ..., v_n)$

for any $v_1, ..., v_n, v \in V$. Let $\{b_1, ..., b_n\}$ be a basis of V, we then can write any $v_j = \sum_{i=1}^n \lambda_i^j b_i$, for some $\lambda_i^j \in \mathbb{R}$ (if V is a vector space over $\mathbb{R}$). So if we fill in any n vectors $v_1, ..., v_n$ of V in f we see that because of the linearity of f we get a linear combination of multiplications of different λ_i^j with $f(b_i^1, b_i^2, ..., b_i^n)$ where $b_i^j \in \{b_1, ..., b_n\}$ (this is too ugly to write out but we just need the f of all possible combinations of the basis vectors. But using the alternating property we know that if $b_i^j = b_i^{j'}$ for $1 \leq i, j, j' \leq n$ we have that $f(b_i^1, ..., b_i^j, ..., b_i^{j'}, ..., b_i^n) = 0$ so we immediately see that none of the b_i^j can be the same. Then the order of which these basis elements are put in f also does not matter as we have the property that if we interchange two arguments we get a minus sign. So all of these $f(b_i^1, b_i^2, ..., b_i^n)$ are linearly dependent on each other. So we can write

$$f(v_1, ..., v_n) = c_1 f(b_1, ..., b_n)$$

with c_1 some constant in $\mathbb{R}$, which is some linear combination of multiplications of λ_i^j. If we have a $g \in \mathrm{Alt}^n(V)$ we can do the same as above and we get

$$g(v_1, ..., v_n) = c_2 g(b_1, ..., b_n)$$

But notice that $g(b_1, ..., b_n), f(v_1, ..., v_n) \in \mathbb{R}$, so

$$g(v_1, ..., v_n) = c_2 g(b_1, ..., b_n) = \frac{c_2 g(b_1, ..., b_n)}{f(v_1, ..., v_n)} f(v_1, ..., v_n) = \frac{c_2 g(b_1, ..., b_n)}{c_1 f(b_1, ..., b_n)} f(v_1, ..., v_n)$$

for all $v_1, ..., v_n \in V$, where we could have taken a function f that has $f(b_1, ..., b_n) \neq 0$. So we see that any $g \in \mathrm{Alt}^n(V)$ can be written as a constant times f, so we see that f is a basis of $\mathrm{Alt}^n(V)$ and therefore it has dimension 1.

Exercise 2.6.3

Let $v, w \in \mathbb{R}^4$ then we can write $v = \sum_{i=1}^{4} \lambda_i e_i$ and $w = \sum_{i=1}^{4} \mu_i e_i$ for some $\lambda_i, \mu_i \in \mathbb{R}$ and $\{e_1, ..., e_4\}$ the canonical basis of $\mathbb{R}^4$. Two independent elements of $\mathrm{Alt}^2(\mathbb{R}^4)$ are then

$$\begin{aligned} f : \mathbb{R}^4 \times \mathbb{R}^4 &\longrightarrow \mathbb{R} \\ (v, w) = \left(\sum_{i=1}^{4} \lambda_i e_i, \sum_{i=1}^{4} \mu_i e_i \right) &\longmapsto \lambda_1 \mu_2 - \lambda_2 \mu_1 \\ g : \mathbb{R}^4 \times \mathbb{R}^4 &\longrightarrow \mathbb{R} \\ (v, w) = \left(\sum_{i=1}^{4} \lambda_i e_i, \sum_{i=1}^{4} \mu_i e_i \right) &\longmapsto \lambda_3 \mu_4 - \lambda_4 \mu_3 \end{aligned}$$

Because the question says "write down"we assume that this needs no explanation.

6 Homework 6

Author: Dax Godding.

2.6.6(b)

Let us write $z = a + bi$ and $w = c + di$. Then we have

$$z^2 - w^3 = (a + bi)^2 - (c + di)^3 = (a^2 - b^2 - c^3 + 3cd^2) + i(2ab - 3c^2 d + d^3).$$

Thus for the function $F : \mathbb{R}^4 \to \mathbb{C}, F(a,b,c,d) = (a^2 - b^2 - c^3 + 3cd^2) + i(2ab - 3c^2d + d^3)$. We know have

$$DF(a,b,c,d) = \begin{pmatrix} 2a & -2b & 3(d^2-c^2) & 6cd \\ 2b & 2a & -6cd & 3(d^2-c^2) \end{pmatrix}.$$

The point $(1,i)$ is now in $\mathbb{R}^4$ equal to the point $(1,0,0,1)$ and thus we obtain

$$DF(1,0,0,1) = \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix}.$$

Note that a parametrization of T is now equal to

$$P : [0.2\pi) \times [0,2\pi) \to T, P(\theta,\phi) = (\cos\theta + i\sin\theta, \cos\phi + i\sin\phi).$$

Now note that $P(0,\frac{\pi}{2}) = (1,i)$ and thus we obtain

$$DP = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & -\sin\phi \\ 0 & \cos\phi \end{pmatrix} \text{ with } DP(0,\frac{\pi}{2}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Finally we know $T_{(1,i)}T = \mathrm{Im}DP(0,\frac{\pi}{2}) = L(e_2,-e_3)$ therefore $Df(1,i) = DF(1,0,0,1)DP(0,\frac{\pi}{2}) = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$.

2.6.6(c)
$\omega(s)$ is alternating because for $a,b \in \mathbb{C}$ we have

$$\omega(s)(a,b) = |s|^2 i(a\bar{b} - \bar{a}b) = |s|^2 i(-(\bar{a}b - a\bar{b})) = -|s|^2 i(b\bar{a} - \bar{b}a) = -\omega(s)(b,a).$$

$\omega(s)$ is also bilinear because for $\mu, \lambda \in \mathbb{R}$ and $a,b,v \in \mathbb{C}$ we have

$$\begin{aligned}\omega(s)(\lambda a + \mu b, v) &= |s|^2 i((\lambda a + \mu b)\bar{v} - \overline{\lambda a + \mu b}v) = |s|^2 i(\lambda a\bar{v} + \mu b\bar{v} - \overline{\lambda a}v - \overline{\mu b}v) \\ &= \lambda|s|^2 i(a\bar{v} - \overline{a}v) + \mu|s|^2 i(b\bar{v} - \bar{b}v) = \lambda\omega(s)(a,v) + \omega(s)(b,v).\end{aligned}$$

$$\omega(s)(v, \lambda a + \mu b) = -\omega(s)(\lambda a + \mu b, v) = -\lambda\omega(s)(a,v) - \omega(s)(b,v) = \lambda\omega(s)(v,a) + \omega(s)(v,b).$$

Also, one can show easily that ω has real values. Thus we conclude that $\omega \in \Omega^2(\mathbb{C})$ is indeed a differential form because $\omega(s) \in \mathrm{Alt}^2(\mathbb{C})$.

2.6.6(d)
We obtain the following for computing $f^*\omega$ at point $p = (1,i)$ to the basis vectors $(e_2,-e_3)$.

$$f^*\omega(1,i)(e_2,-e_3) = \omega(f(1,i))(e_2,-e_3) = \omega(1+i)(e_2,-e_3) = \omega(1+i)(2i,-3) = 2i(-6i-6i) = 24.$$

2.6.7(a)
Let us look at the function

$$\begin{aligned} I :& L(V, \mathrm{Mult}^k(V)) \to \mathrm{Mult}^{k+1}(V) \\ & H \mapsto \{f : V^{k+1} \to \mathbb{R}, f(v_1,\dots,v_{k+1}) \to H(v_1)(v_2,\dots v_{k+1})\}.\end{aligned}$$

Note that the this function I is indeed well-defined because when $f = g \in \text{Mult}^{k+1}(V)$ with

$$f(v_1, \ldots, v_{k+1}) = H(v_1)(v_2, \ldots v_{k+1}) \text{ and } g(v_1, \ldots, v_{k+1}) \to M(v_1)(v_2, \ldots v_{k+1}),$$

then $M = H$. Indeed it is also easy to see that I is bijective because for every $g \in \text{Mult}^{k+1}(V)$ we can write g as follows for some $M \in L(V, \text{Mult}^k(V))$

$$g : V \to \text{Mult}^k(V), g(v_1, \ldots, v_{k+1}) \to M(v_1)(v_2, \ldots v_{k+1}).$$

therefore we have that $I^{-1}(g) = M$ and thus we conclude that I is a isomorphism between $L(V, \text{Mult}^k(V))$ and $\text{Mult}^{k+1}(V)$, which makes the two vector spaces isomorphic.

2.6.8(a)
Let $b_s \in V$ for any $1 \leq s \leq n$ then we have

$$F(b_i)b^i(b_s) = F(b_1)b^1(b_s) + \ldots + F(b_s)b^s(b_s) + \ldots + F(b_n)b^n(b_s) = F(b_s)b^s(b_s) = F(b_s).$$

So we conclude that $F(b_i)b^i$ and F take the same values on the basis $b_1, \ldots, b_n$ and thus we conclude that $F = F(b_i)b^i \in V^*$.

2.6.8(b)
We have the following for $b_s, b_t \in V$ for any $1 \leq s \leq t \leq n$ we have

$$F(b_s, b_t) = F(b^i(b_s)b_i, b^j(b_t)b_j) = F(b_i, b_j)b^i(b_s)b^j(b_t) = F(b_i, b_j)b^i \otimes b^j(b_s, b_t).$$

Again we conclude that $F(b_i, b_j)b^i \otimes b^j$ and F take the same values on the basis $b_1, \ldots, b_n$ and thus we conclude that $F = F(b_i, b_j)b^i \otimes b^j \in \text{Mult}^k(V)$.

2.6.8(c)
It is easy to see that the linear span of the set of tensors $T = \{b^i \otimes b^j | i, j = 1, \ldots, n\}$ contains all tensor products of $\text{Mult}^k(V)$. To show that all $b^i \otimes b^j$ for $i, j = 1, \ldots, n$ are linearly independent because for $c_{ij}b^i \otimes b^j = 0$ we have for all $s, t = 1, \ldots, n$ that

$$0 = c_{ij}b^i \otimes b^j = c_{ij}(b^i \otimes b^j)(b_s, b_t) = c_{ij}b^i(b_s)b^j(b_t) = c_{st}.$$

Thus we conclude that T is indeed linearly independent.

2.6.8(d)
For basis (e_1, e_2) of $\mathbb{R}^2$ we have the following if we can write the inner product as a tensor product

$$0 = \langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = e^1 \otimes e^2(e_1, e_2) = e^1 \otimes e^2(e_2, e_1) = 1.$$
$$1 = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = e^1 \otimes e^1(e_2, e_2) = e^2 \otimes e^2(e_1, e_1) = 0.$$

Both cases gives us a contradiction, thus we conclude that the inner product cannot be written as a tensor product.

7 Homework 7

Author: Bart Eggen (edited).

Exercise 2.7.4

Let $U \subset \mathbb{R}^3$ and $x, y, z \in (\mathbb{R}^3)^*$ the coordinate functions on U.

a.

First let $f \in \Omega^0(U)$, so according to definition we have $f : U \to \mathbb{R}$. From lemma 2.7.1 we know that we can write any $\omega \in \Omega^3(U)$ as

$$\omega = \omega_I dx^I$$

here we sum over all subsets of size 3 of $\{1, 2, 3\}$ and $\omega_I : U \to \mathbb{R}$ for all I . The only subset of size 3 is $\{1, 2, 3\}$ so this means that we have

$$\omega = \omega_{\{1,2,3\}} dx \wedge dy \wedge dz$$

so we see that any $\omega \in \Omega^3(U)$ can be written as $\omega = f dx \wedge dy \wedge dz$, where $f \in \Omega^0(U)$, so indeed $\Omega^3(U)$ can be identified as sending a function $f \in \Omega^0(U)$ to $f dx \wedge dy \wedge dz$.

b.

Again according to lemma 2.7.1 we know that we can write any $\omega \in \Omega^1(U)$ as

$$\omega = \omega_{\{1\}} dx + \omega_{\{2\}} dy + \omega_{\{3\}} dz$$

because the only subsets of size 1 are $\{1\}, \{2\}$ and $\{3\}$. Here $\omega_I : U \to \mathbb{R}$ for all I. But a vector field $F = (F_1, F_2, F_3)$ contains three functions $F_i : U \to \mathbb{R}$ for each of its components. So indeed we can identify $\Omega^1(U)$ as the set of vector fields, because for any vector field $F = (F_1, F_2, F_3)$ we have a corresponding $\omega \in \Omega^1(U)$, which is

$$\omega = F_1 dx + F_2 dy + F_3 dz$$

Also with lemma 2.7.1 we can write any $\omega \in \Omega^2(U)$ as

$$\omega = \omega_{\{1,2\}} dx \wedge dy + \omega_{\{2,3\}} dy \wedge dz + \omega_{\{1,3\}} dx \wedge dz$$

because the only subsets of size 1 are $\{1, 2\}, \{2, 3\}$ and $\{1, 3\}$. Here $\omega_I : U \to \mathbb{R}$ for all I. Notice that, because $dx, dz \in \Omega^1(U)$ we have that

$$dx \wedge dz = (-1)^{1 \cdot 1} dz \wedge dx = -dz \wedge dx$$

So we can also write it as

$$\omega = \omega_{\{1,2\}} dx \wedge dy + \omega_{\{2,3\}} dy \wedge dz - \omega_{\{1,3\}} dz \wedge dx = \omega_{\{1,2\}} dx \wedge dy + \omega_{\{2,3\}} dy \wedge dz + \omega'_{\{1,3\}} dz \wedge dx$$

Also a vector field $F = (F_1, F_2, F_3)$ contains three functions $F_i : U \to \mathbb{R}$ for each of its components. So indeed we can identify $\Omega^2(U)$ as the set of vector fields, because for any vector field $F = (F_1, F_2, F_3)$ we have a corresponding $\omega \in \Omega^2(U)$, which is for example

$$\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

c.

An element $f \in \Omega^0(U)$ is a function $f : U \to \mathbb{R}$, while any element $\omega \in \Omega_1(U)$ can be written as $F_1 dx + F_2 dy + F_3 dz$. So we can see $d : \Omega^0(U) \to \Omega_1(U)$ as sending f to $fdx + fdy + fdz$ which can be seen as the gradient of f, because we have a summation of the "derivatives"towards the different directions. So indeed $d : \Omega_0(U) \to \Omega_1(U)$ corresponds to taking the gradient.

We also know from **b.** that we can identify $\Omega_1(U)$ as the set of vector fields, so for each $\omega \in \Omega_1(U)$ we have a corresponding vector field $F = (F_1, F_2, F_3)$. Also, any $\omega \in \Omega^2(U)$ can be written as $\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. So indeed we can see that d sends a vector field F towards $F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$, which is the curl of this vectorfield.

d.

Let $f \in \Omega^0(U)$, we then know from lemma 2.7.2(1) that $d(df) = 0$. But we saw in **c.** that df corresponds to taking the gradient of f, which is a vectorfield and therefore $d(df)$ corresponds to taking the curl of the vector field df. So from this lemma we see that taking the curl of the gradient of a function is always equal to 0.

Exercise 3.1.1

We have $\gamma : (0, 1) \to C \subset \mathbb{R}^2$ given by $\gamma(t) = (t^2, t^3)$.

a.

We have that $\omega = (x + y)dx + (x - y)^2 dy$. Here $x, y : \mathbb{R}^2 \to \mathbb{R}$ and $dx, dy \in \Omega^1(\mathbb{R}^2)$. So we see that $\omega : \mathbb{R}^2 \to \mathrm{Mult}^1(\mathbb{R}^2) = \mathrm{Alt}^1(\mathbb{R}^2)$. Also the tangent space of $\mathbb{R}^2$ in a point is just $\mathbb{R}^2$, so indeed any $\omega(x) \in \mathrm{Alt}^1(\mathbb{R}^2)$ so we see that $\omega \in \Omega^1(\mathbb{R}^2)$. We also know from the syllabus for $p = (p_1, p_2)$ and $v = (v_1, v_2)$ that $dx(p)(v) = Dx(p)(v) = x(v) = v_1$ and $dy(p)(v) = Dy(p)(v) = y(v) = v_2$. So this gives us

$$\omega(p)(v) = (x(p) + y(p))dx(p)(v) + (x(p) - y(p))^2 dy(p)(v) = (p_1 + p_2)v_1 + (p_1 - p_2)^2 v_2$$

b.

We first calculate the derivative of γ in a direction $a \in (0, 1)$. We have

$$D\gamma(a) = (2a, 3a^2)$$

as it is just the Jacobian if we take the standard basis. Also note that the tangent space of $(0, 1)$ of any point is $\mathbb{R}$, so we take 1 as a basis vector of this (because we want to see what $\gamma^* dx(a)$ does to tangent space vectors). So we get

$$\gamma^* dx(a)(1) = dx(\gamma(a))(D\gamma(a)\dot{1}) = dx(a^2, a^3)(2a, 3a^2) = 2a$$

And we also have

$$\gamma^* dy(a)(1) = dy(\gamma(a))(D\gamma(a)\dot{1}) = dy(a^2, a^3)(2a, 3a^2) = 3a^2$$

c.

Using again the same logic as in **b.** we have

$$\gamma^* \omega(a)(1) = \omega(\gamma(a))(D\gamma(a) \cdot 1) = \omega(a^2, a^3)(2a, 3a^2) = (a^2 + a^3) \cdot 2a + (a^2 - a^3)^2 \cdot 3a^2$$

d.

Using the product rule, the wedge to make it alternating, $dd = 0$ and the property that $d\omega \wedge d\omega = 0$ and $dx \wedge dy = -dy \wedge dx$ as we saw before, we have

$$\begin{aligned} d\omega &= d\left((x+y)dx + (x-y)^2 dy\right) \\ &= d(x+y) \wedge dx + (x+y)ddx + d\left((x-y)^2\right) \wedge dy + (x-y)^2 ddy \\ &= (dx+dy) \wedge dx + 2(x-y)d(x-y) \wedge dy \\ &= dy \wedge dx + 2(x-y)dx \wedge dy \\ &= (2(x-y)-1)\, dx \wedge dy \end{aligned}$$

e.

We have that $\gamma((0,1)) = C$, so from 3.1.1 and the previous parts of the exercise we know that (we leave out the dx in the Riemann integral so it does not become confusing)

$$\begin{aligned} \int_C \omega &= \int_{(0,1)} \gamma^* \omega \\ &= \int_0^1 (x^2 + x^3) \cdot 2x + (x^2 - x^3)^2 \cdot 3x^2 \\ &= \int_0^1 2x^3 + 2x^4 + 3x^6 - 6x^7 + 3x^8 \\ &= \frac{1}{2} + \frac{2}{5} + \frac{3}{7} - \frac{3}{4} + \frac{1}{3} = \frac{383}{420} \end{aligned}$$

Exercise 3.1.2

From symmetry arguments, it follows that the integral should be zero independent of the parametrization.

We want to calculate $\int_{S^2} z dx \wedge dy$ using spherical coordinates. These coordinates are given by

$$\begin{aligned} \gamma_1 : [0,\pi] \times [0, 2\pi) &\longrightarrow S^2 \subset \mathbb{R}^3 \\ (\theta, \phi) &\longmapsto \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \end{aligned}$$

The derivative of this map is given by (in the standard basis)

$$D\gamma_1(\theta,\phi) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\phi\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

We also have, using the definition and the previous question (to see what dx and dy do)

$$\begin{aligned} \gamma_1^*(z dx \wedge dy)(\theta,\phi)(e_1, e_2) &= (z dx \wedge dy)(\gamma(\theta,\phi))(D\gamma_1(\theta,\phi)e_1, D\gamma_1(\theta,\phi)e_2) \\ &= \cos\theta \cdot (\cos\theta\cos\phi \cdot \sin\phi\cos\phi) = \cos^2\theta\cos^2\phi\sin\phi \end{aligned}$$

So now with lemma 3.1.1 we have

$$\int_{S^2} z dx \wedge dy = \int_{[0,\pi]\times[0,2\pi)} \gamma_1^*(z dx \wedge dy) = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \cos^2\theta \cos^2\phi \sin\phi d\phi d\theta = \frac{\pi}{2} \cdot 0 = 0$$

We will now calculate the integral using cartesian coordinates. Let us parametrize the S^2 with

$$\phi_1(x, y) = (x, y, \sqrt{1 - x^2 y^2}), \phi_2(x, y) = (x, y, -\sqrt{1 - x^2 y^2}),$$

both maps with domain D. Calculating the pullback $\phi_i^* \omega_{(x,y)}(e_1, e_2) = \omega_{\phi_i(x,y)(D\phi_i(e_1), D\phi_i(e_2))}$ for ϕ_1, ϕ_2 gives us $\sqrt{(}1 - x^2 - y^2)$ and $-\sqrt{1 - x^2 - y^2}$ for i=1 and i=2 respectively. Now we get

$$\int_{S^2} \omega = \int_D \sqrt{1 - x^2 y^2} - \int_D \sqrt{1 - x^2 - y^2} = 0. \tag{3}$$