Groningen Topology Seminar

Exercise sheet 7

5th December 2019

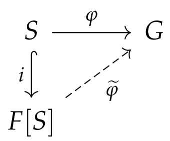
1. (Def) Recall that given a set S, the **free group** generated by S, denoted $F(S)$, is the set of finite words $s_1^{\varepsilon_1} \cdots s_r^{\varepsilon_r}$ of elements of S with exponents $\varepsilon_i = \pm 1$, where there is not a pair of joint letters $s^{\varepsilon} s^{-\varepsilon}$. It is a group with the concadenation of words (simplifying if there is a pair $s^{\varepsilon} s^{-\varepsilon}$) and the unit element is the empty word). Observe that there is a natural map of sets $i : S \hookrightarrow F(S)$ sending $s \in S$ to the word with only one letter s (with power +1).

   If $S = \{x_1, \dots, x_n\}$, then $F(S)$ is usually written $F(x_1, \dots, x_n)$.

2. Show that every finitely generated group G is a quotient of a free group $F(x_1, \dots, x_n)$.

   An isomorphism
   $$G \cong \frac{F(x_1, \dots, x_n)}{< r_1, \dots, r_k >}$$
   is called a **presentation** of G. Here r_i are **relations** generating a certain subgroup.

3. Show that the free group has the following universal property[1]: given a group G, and a map of sets $f : S \longrightarrow G$, there exists a unique group homomorphism $\widetilde{f} : F(S) \longrightarrow G$ such that $\widetilde{f} \circ i = f$.



4. (Def) Let G_1, G_2 be groups. The **free product** of G_1 and G_2 is the set $G_1 * G_2$ of finite words $g_1 \cdots g_r$ of elements of $G_1 \coprod G_2$, where there are not two consecutive elements of the same group, and neither of the elements is the unit element of G_1 or G_2. Again, it is a group under concadenation (where if two letters gg' belonging to the same group appear together, they are considered as a single element of the group). The unit element is the empty sequence.

5. Show that if S, S' are sets, then $F(S) * F(S') \cong F(S \coprod S')$.

6. (Def) Let G_0, G_1, G_2 be groups and consider a pair of group homomorphisms
   $$\delta_1 : G_0 \longrightarrow G_1 \qquad , \qquad \delta_2 : G_0 \longrightarrow G_2.$$
   The **amalgamated product** of G_1 and G_2 over G_0 (with respect to the morphisms δ_1, δ_2) is the group
   $$G_1 *_{G_0} G_2 := \frac{G_1 * G_2}{< \delta_1(g_0) = \delta_2(g_0) >}.$$
   The quotient stands for the smallest normal subgroup generated by the elements $\delta_1(g_0)\delta_2(g_0)^{-1}$, for $g_0 \in G_0$.

7. Show that $G_1 *_1 G_2 \cong G_1 * G_2$, where 1 is the trivial group and δ_i is the unique group homomorphism $1 \longrightarrow G_i$.

8. Let $G_0 = F(u)$, $G_1 = F(x)$ and $G_2 = F(y)$. Let $\delta_1(u) := x$ and $\delta_2(u) := y^2$. Compute $G_1 *_{G_0} G_2$.

[1] If you know category theory, this says that the free group construction is left adjoint to the forgetful functor $\mathsf{Grp} \longrightarrow \mathsf{Set}$.