Groningen Topology Seminar

Exercise sheet 5

7th November 2019

1. Show that $\mathbb{R}^n - \{0\}$ is homotopy equivalent to S^{n-1}.
2. Show that if Y is contractible, then any two maps $f, g : X \longrightarrow Y$ are homotopic.
3. (Ruben's exercise)
   (a) If $f, g : X \longrightarrow Y$ are homeomorphisms, then $f(X) \cong g(X)$? And if they are homotopic maps, then $f(X) \simeq g(X)$?
   (b) If f is homotopic to a constant map, is $f(X)$ contractible?
   (c) If $f : X \longrightarrow Y$ is injective, so is $f_* : \pi(X, p) \longrightarrow \pi(Y, f(p))$? What about surjective?
   (d) If $f : X \longrightarrow Y$ is the constant map, is $f_* : \pi(X, p) \longrightarrow \pi(Y, f(p))$ the zero map?
4. Let us give a topological proof of the

   Theorem 1 (Fundamental of Algebra). *Any polynomial* $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *of degree* $n > 0$ *with complex coefficients has a complex root.*

   (a) Show that every group homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ must be multiplication by some integer.
   (b) Show that if $f : S^1 \longrightarrow S^1, f(z) = z^n$, then

   $$f_* : \pi(S^1) \cong \mathbb{Z} \longrightarrow \mathbb{Z} \cong \pi(S^1)$$

   is multiplication by n.
   (c) Compute the group homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by a constant map $S^1 \longrightarrow S^1$.
   (d) (The actual theorem starts here) Suppose that $p(x)$ has no roots. Consider the maps $f, g : S^1 \longrightarrow \mathbb{C} - \{0\}$ defined by $f(z) = z^n$ and $g(z) = a_0$. Show that

   $$H : S^1 \times I \longrightarrow \mathbb{C} - \{0\} \qquad , \qquad H(z, s) := \sum_{i=0}^{n} a_i s^{n-i} ((1-s)z)^i$$

   is a (well-defined!) homotopy between f and g.
   (e) Use the previous items of the exercise to get a contradiction!