# Groningen Topology Seminar

Exercise sheet 3

24th October 2019

## More general topology

1. **(Glueing lemma)** Let $X, Y$ be topological spaces and let $F_1, F_2 \subset X$ closed subsets such that $X = F_1 \cup F_2$. Suppose we have continuous maps $f_1 : F_1 \longrightarrow Y$, $f_2 : F_2 \longrightarrow Y$ which coincide in $F_1 \cap F_2$. Show that the "glueing" function $f_1 \cup f_2 : X \longrightarrow Y$ is continuous.

2. **(Lebesgue covering lemma)** Let $K$ be a compact metric space, and let $\{U_i\}$ be an open cover of $K$. Show that there exists a positive number $\delta > 0$ (called the **Lebesgue number** of the cover) such that for every subset $A \subset K$ of diameter less than $\delta$, it holds that $A \subset U_i$ for some open subset $U_i$ of the cover.

## Fundamental group

In the following, the symbol " $\simeq$ " will denote "homotopic".

3. Show that any two paths $\alpha, \beta : I \longrightarrow \mathbb{R}^n$ with same endpoints are homotopic.

   Show that any loop in $\mathbb{R}^n$ at 0 is homotopic to the constant loop.

   Conclude that $\pi_1(\mathbb{R}^n, 0) \cong 0$.

4. Let $X$ be a space and let $\gamma$ be a path with $\gamma(0) = p, \gamma(1) = q$. Show that $\gamma$ induces an isomorphism

$$\pi_1(X, p) \xrightarrow{\cong} \pi_1(X, q) \qquad , \qquad [\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma].$$

   Conclude that if a topological space is **path-connected** (that is, every pair of points can be connected by a path) then the fundamental group of a space is independent of the basepoint.

5. Let $f : X \longrightarrow Y$ be a continuous map. Show that if $\alpha \simeq \alpha'$, then $f \circ \alpha \simeq f \circ \alpha'$.

   Show that for any two loops $\alpha, \beta$ it holds that $(f \circ \alpha) \cdot (f \circ \beta) \simeq f \circ (\alpha \cdot \beta)$.

6. Show that the homotopy of paths has the following cancellation property: if $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$, and $\beta \simeq \beta'$, then $\alpha \simeq \alpha'$.