Groningen Topology Seminar

Exercise sheet 2

17th October 2019

Topological properties

A topological space (X, τ) is **Hausdorff** if every pair of different points have disjoint (open) neighbourhoods.

1. Show that every metric space is Hausdorff. Is $(\mathbb{R}, \tau = \{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}\})$ Hausdorff?

   A topological space (X, τ) is **compact** if every open cover of X has a finite subcover.

2. Is $(X, \tau_{\text{trivial}})$ compact? What about $(X, \tau_{\text{discrete}})$?
3. Show that if X is compact, and $A \subset X$ is closed, then A is also compact (with the subspace topology).
4. Show that if X is Hausdorff, and $A \subset X$ is compact, then A is closed (with the subspace topology).
5. Show that if $f : (X, \tau) \longrightarrow (Y, \tau')$ is a continuous bijection with X compact and Y Hausdorff, then f is homeomorphism. (*Hint:* Use ex. 3 and 4!)

Quotient topology

Recall that given a topological space (X, τ) and an equivalence relation $\sim$ on X, the **quotient topology** on $X/\sim$ has open subsets $A \subset X/\sim$ such that $p^{-1}(A) \subset X$ is open, where $p : X \longrightarrow X/\sim$.

6. Construct the torus $\mathbb{T} := S^1 \times S^1$ out of a sheet of paper, writing down the identifications.
7. Show that every quotient space of a space with the trivial topology also has the trivial topology. Show that the same happens with the discrete topology.
8. Let (X, τ) be a topological space and let $\sim$ be an equivalence relation on X. Show that a map $g : (X/\sim, \tau_q) \longrightarrow (Y, \tau')$ is continuous if and only if $g \circ p : (X, \tau) \longrightarrow (Y, \tau')$ is continuous.
9. (**Universal Property of the Quotient Topology**) Let $f : (X, \tau) \longrightarrow (Y, \tau')$ be a continuous map and let $\sim$ be an equivalence relation on X. Show that

   $$[x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)] \iff \text{There exists } \bar{f} : X/\sim \longrightarrow Y \text{ continuous such that } \bar{f} \circ p = f.$$

   The **projective plane** is the quotient space $\mathbb{RP}^2 := (\mathbb{R}^3 - \{0\})/\sim$ under the equivalence relation $x \sim y \iff y = \lambda x$ for some $0 \neq \lambda \in \mathbb{R}$.

10. Show that $\mathbb{RP}^2$ is homeomorphic to the sphere $S^2/\sim$ where we identify antipodal points, $x \sim -x$.
11. What space do we obtain if we take a Moebius strip and we identify its boundary circle to a point?