Topology Seminar

Exercise sheet 1

10th October 2019

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Check that the map

$$d : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R} \qquad , \qquad d(n,m) := \begin{cases} n+m, & n \neq m \\ 0, & n = m \end{cases}$$

   is a metric. If $B(x,r)$ denotes the open ball with centre at $x \in \mathbb{N}$ and radius $r > 0$, compute $B(2,10)$ and $B(10,2)$.

2. Let $X := \{N, S, E, W\}$ be a set with four points. Check that

$$\tau := \Big\{\{N,S,E,W\}, \{N,S,E\}, \{N,S,W\}, \{N,S\}, \{N\}, \{S\}, \emptyset\Big\}$$

   is a topology over X. This topological space is called the **pseudocircle**.

3. Describe the neighbourhoods of a point $x \in \mathbb{R}$ in the topological space $(\mathbb{R}, \tau = \{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}\})$. Do the same with $(\mathbb{R}, \tau_{\text{cofinite}})$.

4. Show that the discrete topology over a set X is induced by a metric in X. Moreover, if $\#X > 1$, show that the trivial topology over X is not induced by any metric in X.

5. Let $(X,\tau), (Y,\tau')$ be topological spaces. Show that any constant map $(X,\tau) \longrightarrow (Y,\tau')$ is continuous.

6. Show that any map $f : (X, \tau_{\text{discrete}}) \longrightarrow (Y, \tau)$ from a discrete space to any other space is continuous.

7. Show that any map $f : (X,\tau) \longrightarrow (Y, \tau_{\text{trivial}})$ from any space to a space with the trivial topology is continuous.

8. Show that a map $f : (X,\tau) \longrightarrow (Y,\tau')$ is a homeomorphism if and only if it is bijective, continuous and an **open map** (this means that for any $O \in \tau$ we have that $f(O) \in \tau'$).

9. Consider over the closed interval $[0, 2\pi]$ the equivalence relation

$$0 \sim 2\pi \qquad , \qquad x \sim x \quad \text{if } x \neq 0, 2\pi.$$

   Endow $[0,2\pi]/\sim$ with the quotient topology, and consider

$$S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2,$$

   endowed with the subspace topology of $\mathbb{R}^2$.
   Show that the map

$$f : [0,2\pi]/\sim \longrightarrow S^1 \qquad , \qquad f(x) := (\cos x, \sin x)$$

   is an homeomorphism.