EIGHT FACES OF THE POINCARÉ HOMOLOGY 3-SPHERE

R. C. Kirby

Department of Mathematics
University of California
Berkeley, California

M. G. Scharlemann[1]

Department of Mathematics
University of California
Santa Barbara, California

We give eight different descriptions of the Poincaré homology sphere, and show that they do define the same 3-manifold. The definitions are: (1) plumbing on the E_8 *graph, (2) surgery on the* E_8 *link, (3) the link of the singularity* $z_1^2 + z_2^3 + z_3^5 = 0$*, (4)* S^3/I^* *where* I^* *is the binary icosahedral group, (5) the dodecahedral space, (6) the Seifert bundle, (7) surgery on the trefoil knot, (8) the p-fold cover of the (q,r)-torus knot, for* $\{p,q,r\} = \{2,3,5\}$.

The dodecahedral space of Poincaré was established long ago as a manifold of unusual interest, both because it was the first example of a homology sphere which is not a sphere and also because it lies in a class of three manifolds closely related to the Platonic solids. Interest in the manifold has increased in recent

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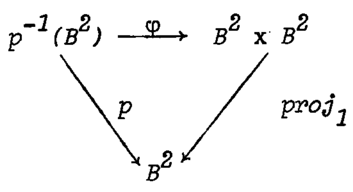
years because of its surprisingly diverse applications to problems in topology (see e.g. [19, §2], [20], [16], [17], [21]). Part of the explanation for its usefulness is the large number of ways, discovered over the years, to describe the dodecahedral space. It is our aim in this paper to collect the most useful of these and verify at an elementary level that all do define the same 3-manifold. This paper arose from seminar notes in 1973 (and we thank L. Siebenmann for a substantial contribution to that seminar). We apologize for the untimely delay in appearance of this exposition, and remind the reader that since 1973, two excellent works, [12] and [14], have appeared which include parts of this paper.

I. EIGHT DESCRIPTIONS

Description 1 (Plumbing). Let $p : T^4 \to S^2$ be the contangent disk bundle over $S^2 = CP^1$ (this is just the tangent disk bundle with the opposite orientation so that the Euler characteristic is -2). Over any cell B^2 in S^2 the bundle is trivial so there is a commutative diagram



where φ is a diffeomorphism.

Two copies T_1 and T_2 of T can be "plumbed" together by identifying, for any $(x,y) \in B^2 \times B^2$, the points $\varphi_1^{-1}(x,y)$ and $\varphi_2^{-1}(y,x)$. The fibers of the first bundle over B^2 correspond to trivial sections of the second bundle over B^2.

Let P^4 be the result of plumbing together 8 copies of T as follows:

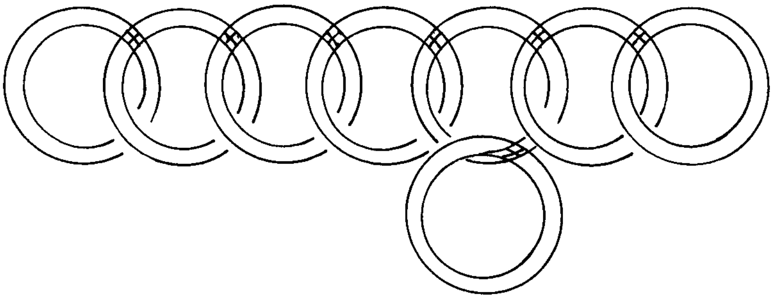


Fig. 1.

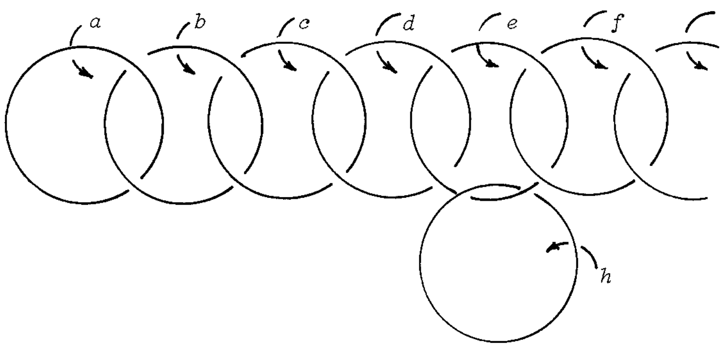


Fig. 2.

After rounding corners, P becomes a smooth 4-manifold. The first description of the dodecahedral manifold will be ∂P^4. In some of the future descriptions, we will recover not only ∂P^4, but P^4 as well.

Description 2 (Surgery on a link). Consider the link Λ of 8 circles in S^3, drawn in Figure 2. Each circle can be assumed planar in $R^3 = S^3 - \infty$, so each has an obvious trivialization of its normal disk bundle (choose one normal vector field in the plane, one orthogonal to the plane). The trivialization τ which we choose, however, is one obtained from the first by rotating the normal disk at each $\theta \in S^1$ by an angle 2θ. Attach $B^2 \times B^2$ to B^4 by $\tau : \partial B^2 \times B^2 \to S^3 = \partial B^4$. Since we may regard B^4 as the trivial 2-disk bundle over a 2-disk whose boundary is the attaching circle, the result is a 2-disk bundle over S^2. Since the framing chosen differs from the standard framing by 2 full left handed twists, the bundle has Euler characteristic -2, and so is the co-tangent bundle of $S^2 = CP^1$.

Adjoin 8 copies of $B^2 \times B^2$ to B^4, one to each circle in Λ using the trivialization τ. The boundary of the resulting manifold is the second description of ∂P^4. In fact, since a pair of linking circles in Λ bound 2-disks in B^4 which intersect at just one point, it may be seen by inspection that descriptions 1 and 2 are equivalent, indeed that the 4-manifold just described is P^4.

With this description it is easy to compute $\pi_1(\partial P^4)$ using the calculus of Crowell and Fox [2]. The group $\pi_1(S^3 - \Lambda)$ is generated by loops around each circle a, b, c, d, e, f, g, h with relations for each crossing, $ab = ba$, $bc = cb$, $cd = dc$, $de = ed$, $ef = fe$, $fg = gf$, $eh = he$. The 8 copies of $B^2 \times \partial B^2$ attached to $S^3 - \Lambda$ provide 8 more relations $1 = a^2b = ab^2c = bc^2d = cd^2e = de^2fh = ef^2g = fg^2 = eh^2$. By substitution $e = a^5 = g^3 = h^{-2}$ and $h^{-1} = ag$, so we have generators a and g with $a^5 = g^3 = (ag)^2$.

This group $\{x,y;\ x^3 = y^5 = (yx)^2\}$, has an independent history, and is known in the literature (for reasons which will become clear) as the binary icosahedral group I^*. I^* is the only finite group which can occur as the fundamental group of a homology 3-sphere [8].

Description 3 (Link of a singularity). Let $f : C^3 \to C$ be the complex polynomial $f(z_1,z_2,z_3) = z_1^2 + z_2^3 + z_3^5$. $f^{-1}(0)$ is a complex variety which is non-singular except where $\partial f/\partial z_j = 0$ for all $j = 1,2,3$. Evidently the only singular point is the origin $z_1 = z_2 = z_3 = 0$. The intersection of the unit 5-sphere about the origin with this variety will also be shown to be ∂P^4.

Description 4 (The quotient S^3/I^*). The icosahedron is a regular solid with twenty faces, thirty edges and twelve vertices. It is the dual complex to the dodecahedron. The group I of isometries of the icosahedron (or dodecahedron) centered at the origin is naturally a subgroup of $SO(3)$, the group of orthogonal rotations of R^3.

Let $SU(2)$, the unitary transformations of C^2, act on C^2 on the right, that is, if $u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a\bar{a} + b\bar{b} = 1$, then $u(z,w) = (z,w)\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$. This action of $SU(2)$ on C^2 commutes with complex multiplication, taking lines to lines, so it defines an action on $CP^1 = S^2 = C^1 \cup \infty$. If $u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$, then for $z \in C^1 \cup \infty$, $u(z) = (z,1)\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = (az - \bar{b}, bz + \bar{a}) = \dfrac{az - \bar{b}}{bz + \bar{a}}$. Hence u gives a linear fractional transformation of $C^1 \cup \infty$. If we identify $C^1 \cup \infty$ with S^2 by stereographic projection, these transformations map onto $SO(3)$. This map $q : SU(2) \to SO(3)$ defines a covering projection which is 2-fold since $q^{-1}(\text{identity}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. (topologically this is the map $S^3 \to RP^3$.)

The lift of I to $SU(2)$ is denoted I^*; we will later show that I^* is the group $\pi_1(\partial P^4)$ calculated above. Since $SU(2)$ considered as a map $R^4 \to R^4$ preserves distance from $\{0\}$, $SU(2)$ acts on S^3. (Later we will show that in fact $SU(2)$ is S^3.) The quotient of S^3 by I^* is the fourth description of ∂P^4. Indeed, we will show there is a homeomorphism $C^2/I^* \to f^{-1}(0)$ above which is biholomorphic off of zero.

Description 5 (Poincare's). The dual of the icosahedron, the dodecahedron, is a regular solid with twelve faces, thirty edges and twenty vertices (see for example, [3, p. 11]). Identify opposite faces of the dodecahedron by the map which pushes each face through the dodecahedron and twists it $2\pi/10 = 36^\circ$ about the axis of the push in the direction of a right-hand screw. This identification is consistent along the edges (see [18]) and the quotient space is a 3-manifold (this requires some checking along the edges). This 3-manifold is ∂P^4.

Description 6 (Seifert bundle). ∂P is a Seifert bundle over S^2 with three exceptional fibers of Seifert invariant $(2,1)$, $(3,1)$ and $(5,1)$ and cross-section obstruction -1. Equivalently ∂P may be obtained by surgery with appropriate framings on any three "anti-Hopf" circles in S^3.

Here we outline what this means (see [13], Chapt. 1). Let M be an oriented 3-manifold with a smooth circle action. Each orbit α has a neighborhood diffeomorphic to $S^1 \times B^2$ (α corresponds to $S^1 \times 0$), with slices $s \times B^2$, $s \in S^1$, being taken to slices. The orbit is principal if $S^1 \times b$ is also an orbit for all $b \in B^2$. The orbit is exceptional with Seifert invariant $(n,1)$ if its neighborhood could be obtained from the principal orbit case by cutting $S^1 \times B^2$ at some slice $s \times B^2$, rotating the slice $2\pi/n$, and then gluing back together (assume the orbit followed by the slice gives the orientation of M). Thus an orbit near an exceptional orbit goes n times parallel to the exceptional orbit, and once around it; S^1 pushes a point *on* the exceptional orbit n times around the orbit. If $d_n : S^1 \times \partial B^2 \to S^1 \times \partial B^2$ is a diffeomorphism represented by the matrix $\begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}$, then $d_n(S^1 \times b)$ is the typical orbit near an exceptional orbit. The action of $\theta \in S^1$ near the orbit is given by $\theta(d_n(s,b)) = d_n(\theta \cdot s, b)$.

Denote the quotient space M/S^1 by $\overline{M}$. If there are only principal and exceptional orbits, then $\overline{M}$ is an oriented 2-manifold and away from the exceptional orbits M is an S^1 fiber bundle over $\overline{M}$. A cross-section to the action on the boundary of a tubular neighborhood of an $(n,1)$ orbit is given by $d_n(s \times \partial B^2)$. There is an obstruction in $H^2(\overline{M}, \pi_1(S^1)) = Z$ to extending these cross-sections to a cross-section to the circle action over all of M-(exceptional orbits). Choose a sigh for this obstruction as follows:

Let (j,k) denote a path going j times around S^1 and k times around ∂B^2. If, for some exceptional orbit, we choose as a cross-section not $d_n(0,1)$ but $d_n(c,1)$, we say the obstruction changes by c. In particular, for some c, the obstruction to extending $d_n(-c,1)$ vanishes; we call c the cross-section obstruction.

The cross-section obstruction may also be defined as follows. Let $\alpha : S^1 \times B^2 \hookrightarrow M$ be a tubular neighborhood of a *principal* orbit $\alpha(S^1 \times 0)$ so that the action at $\theta \in S^1$ is given by $\theta(s,b) = (\theta \cdot s, b)$. A cross-section for the action is $\alpha(0,1)$ then c is the integer such that $\alpha(-c,1)$ extends to a cross-section of all other principal orbits, a cross-section which coincides with $d_n(0,1)$ near the exceptional orbits.

As an example, let $M = S^3$ be the unit sphere around the origin in C^2. One action of the circle on S^3 is given by $\lambda(z,w) = (\lambda z, \bar{\lambda} w)$, $\lambda \in S^1 \subset C$, $(z,w) \in S^3 \subset C^2$. All the orbits are principal; indeed the quotient map is the "anti-Hopf" fibration $\bar{H} : S^3 \to S^2$, which is conventionally oriented so that the Euler class is $+1$. (This convention is motivated by the theory of complex manifolds, in which the natural action of S^1 on D^4, given by $\lambda(z,w) = (\lambda z, \lambda w)$, $(z,w) \in D^4 \subset C^2$, may be lifted to an action on the Hopf bundle by "blowing up" the origin in D^4 replacing the origin by a 2-sphere whose normal bundle has Euler class -1).

In general for M a bundle over $\bar{M}$ with no exceptional orbits and cross-section obstruction c, the Euler class is $-c$. Here, in particular, is how to see that the cross-section obstruction for the anti-Hopf circle action on S^3 is -1. Regard S^3 as the union $(S^1 \times B^2)_1 \cup_f (S^1 \times B^2)_2$ of two solid tori by a homeomorphism $f : (S^1 \times \partial B^2)_1 \to (S^1 \times \partial B^2)_2$ whose matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $(S^1 \times B^2)_i$ have zero-section $S^1 \times \{0\}$ corresponding to the axis $w = 0$ for $i = 1$ and $z = 0$ for $i = 2$. Let $\theta \in S^1$ act on $S^1 \times B^2$ by $\theta(s,b) = (\theta s, b)$. The anti-Hopf action of the circle on S^3, restricted to $(S^1 \times B^2)_1$, is then $\alpha\,\theta\,\alpha^{-1}$, where $\alpha : S^1 \times B^2 \to (S^1 \times B^2)_1$ is the homeomorphism whose matrix is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Then $f\alpha(1,1) = (0,1)$ which extends to the cross-section $(\ \times B^2)_2$ over $(S^1 \times B^2)_2$. Hence $c = -1$.

To construct ∂P^4, remove three orbits from S^3 (labeled $\alpha_2, \alpha_3, \alpha_5$) and sew them back in using d_2, d_3 and d_5. More precisely, construct

$$(S^3 - (\alpha_2 \cup \alpha_3 \cup \alpha_5)) \cup_{g_2} S^1 \times B^2 \cup_{g_3} S^1 \times B^2 \cup_{g_5} S^1 \times B^2$$

where

$$g_i : S^1 \times (B^2 - 0) \to S^1 \times (B^2 - 0)$$

is $d_i \times id_{(0,1]}$ (we consider $B^2 - 0$ to be $\partial B^2 \times (0,1]$) and the domain of g_i is identified with a neighborhood of α_i, $i = 2,3,5$ by taking $S^1 \times b$ to an anti-Hopf circle for each $b \in B^2 - 0$. See Figure 3.

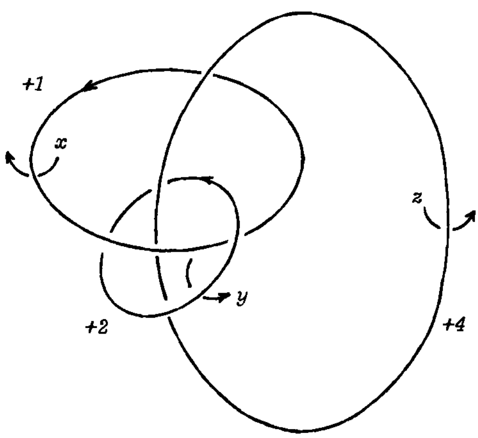


Fig. 3.

Clearly this describes the above Seifert bundle, and, simultaneously, a surgery on a link in S^3. Once again it is possible to calculate the fundamental group knot-theoretically. Let L be the link of 3 Hopf circles (Figure 3); then

$\pi_1(S^2 - L) = \{x,y,z \mid x = y^{-1}z^{-1}xzy,\ y = z^{-1}x^{-1}yxz,\ z = x^{-1}y^{-1}zyx\}$

Each surgery kills the element corresponding to $d_n^{-1}(s, \partial B^2)$, which is a curve going once along α_n and winding around α_n n times compared to an anti-Hopf circle. Thus in Figure 2, the curves winding around n-1 times are killed and we add the relations

$1 = x^{-1}zy = y^{-2}xz = z^{-4}yz$. Then $x = zy$ and $x^2 = y^3 = z^5$, so the fundamental group is again I^*.

Description 7 (surgery on the trefoil knot). Surgery on the left handed trefoil knot L (Figure 4) with framing -1 gives ∂P^4 (this trefoil knot is called left handed because the crossings correspond to a left handed screw). A knot bounds a smooth orientable surface in S^3, which determines a normal vector field to the knot (tangent to the surface) and hence a framing (or trivialization) for the normal bundle. This is the zero framing, and framing n comes from twisting the 0-framing n times in a right handed direction. If we push the trefoil knot off itself using the framing, we get a curve homotopic to the dotted curve c in Figure 4. A presentation for $\pi_1(S^3 - L)$ is $\{a,b,c \mid ab = bc = ca\}$. Surgery kills the class represented by c, so we add the relation $baca^{-2} = 1$. Since $c = b^{-1}ab = aba^{-1}$, we have $(ab)^3 =$ $(ab)(\bar{b}c)(ca) = (a\bar{b}c)^2 = (a^2\bar{b})^2 = a^2(\bar{b}a^2\bar{b}a^{-3})a^3 = a^2(\bar{b}aca^{-2})a^3 = a^5$ so the group is $I^* = \{a,\bar{b} \mid a^5 = (a\bar{b})^3 = (a^2\bar{b})^2\}$.

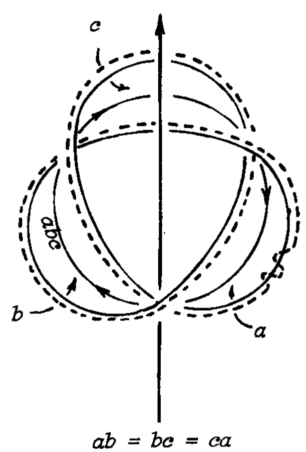


Fig. 4.

Description 8 (A branched cover). ∂P^4 is the 5-fold branched covering over the right handed trefoil knot (= (2,3) torus knot). Similarly it is the 2-fold branched covering of the (3,5) torus knot and the 3-fold branched covering of the (2,5) torus knot.

Here is a brief description of the n-fold branched cover of a knot K in S^3. K has a trivial normal bundle. We will show that for some trivialization of the normal bundle, $T : S^1 \times B^2 \to S^3$, $T(S^1 \times 0) = K$, there exists a map $f : S^3 - K \to S^1$, unique up to homotopy, such that $fT|S^1 \times \partial B^2 = p_2 : S^1 \times \partial B^2 \to \partial B^2$. Let E denote the total space of the normal circle bundle of K.

Recall that for any space X, there is a natural isomorphism $[X,S^1] = [X,K(Z,1)] \cong H^1(X,Z)$. Since, for F a fiber of the normal circle bundle to K, inclusion induces an isomorphism $H^1(S^3 - K;Z) \cong H^1(F;Z)$, it also induces an isomorphism $[S^3 - K, S^1] \to [F, S^1]$. Thus a generator f of $[S^3 - K, S^1]$ carries F to S^1 by a degree one map. But varying the framing $T : S^1 \times B^2 \to S^3$ changes the degree of the composite $(S^1 \times b) \xrightarrow{T} E \xrightarrow{f} S^1$, $b \in \partial B^2$, by multiples of (degree $f|F$) $= 1$; hence we may choose T so that $(S^1 \times b) \xrightarrow{T} E \xrightarrow{f} S^1$ is zero. But the map $p_2 : S^1 \times \partial B^2 \to \partial B^2 \cong S^1$ is also of degree one on $F = (s \times \partial B^2)$ and degree zero on $(S^1 \times b)$. Since the two maps are homotopic on the 1-skeleton of E, they are homotopic on E. Thus we may take T so that $fT|S^1 \times \partial B^2 = p_2$.

To define the n-fold branched covering space Σ_n of K, let V be the bundle over $S^3 - K$ induced by f and the n-fold covering of S^1;

$$\begin{array}{ccc} V & \longrightarrow & S^1 \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle n} \\ S^3 - K & \xrightarrow{f} & S^1 \end{array}$$

The end of V is homeomorphic to $S^1 \times S^1 \times R$, so we can sew K back in to obtain the manifold Σ_n. The projection ν defines a map $\Sigma_n \xrightarrow{\nu} S^3$ which is a homeomorphism on $\nu^{-1}(K)$ and an n-fold covering elsewhere.

II. EQUIVALENCE OF THE DESCRIPTIONS

We will prove the following equivalences:

$$\begin{array}{ccccccccc} (5) & \Leftrightarrow & (4) & \Leftrightarrow & (3) & \Leftrightarrow & (6) & \Leftrightarrow & (1) \\ & & & & \Updownarrow & & \Updownarrow & & \Updownarrow \\ & & & & (8) & \Leftrightarrow & (7) & \Leftrightarrow & (2) \end{array}$$

While several arguments are long, they are designed to be self-contained. No deep theorems are required.

A few words are necessary about orientations. A complex manifold has a unique orientation ($SU(n)$ is connected) and we use this fact to determine a preferred orientation for ∂P^4. The variety $z_1^2 + z_2^3 + z_3^5 = 0$ is the cone on ∂P^4 and if we require (traditionally) that the first vector of its unique orientation be an outward pointing normal to ∂P^4, then we have oriented ∂P^4. If the singularity is resolved, we get the complex manifold int P^4 which corresponds to plumbing disk bundles with Euler characteristic -2.

If two complex linear subspaces in C^n intersect at a point, then together they must give the unique orientation of C^n, so algebraically their intersection must be $+1$. Thus the Hopf circles in S^3 (which are the intersections of complex lines in C^2 with S^3), must have linking number $+1$  ; also R^3 has the usual right handed orientation. The variety $z_1^2 + z_2^3 = 0$ (or $z_1^2 = z_2^3$) in C^2 meets S^3 in the right handed trefoil knot ($2,3$ torus knot).

On the other hand, $-\partial P^4$ bounds a complex manifold, the handle body obtained by attaching a 2-handle to B^4 along the right handed trefoil knot with framing $+1$. This complex manifold union P^4 is $CP^2 \# \overset{8}{\#}(-CP^2)$.

The most useful reference is [13] which contains proofs (buried in more general theorems) of the equivalences *(1)*$\Longleftrightarrow$*(3)*$\Longleftrightarrow$*(4)*$\Longleftrightarrow$*(6)*; our proof of *(3)*$\Longleftrightarrow$*(8)* is taken from [12], which also contains a somewhat different proof of *(3)*$\Longleftrightarrow$*(4)*. A proof of *(5)*$\Longleftrightarrow$*(4)*$\Longleftrightarrow$*(3)*$\Longleftrightarrow$*(1)* for a more general class of manifolds can be found in [4]. [14] contains equivalences *(2)*$\Longleftrightarrow$*(7)*$\Longleftrightarrow$*(8)*.

Equivalence of descriptions (4) and (5): First we describe how to picture S^3/I^*, then provide a proof that indeed S^3/I^* is the dodecahedron with opposite sides identified. Imagine the following circular chain of dodecahedra: place one dodecahedron on one face on the table, and then place nine more on it to form a tower with each dodecahedron rotated $\pi/5$ around the vertical axis compared to the one just below it; then identify top and bottom. Take another copy of this circular chain and place it adjacent to the first, at a slant of 36°, and winding once around the first, like a pair of Hopf circles. In this way wind five circular chains about the first one. In R^3 these do not fit perfectly together, but in S^3 they do. Note that it makes no difference which way they wind. Take two copies of this and sew them together the way one sews together solid tori to get S^3.

Thus S^3 is decomposed into 120 dodecahedra whose centers can be taken to be the elements of I^*. These elements permute the dodecahedra; in particular there is an element of I^* which pushes our original dodecahedron up one in the tower, identifying bottom and top of the dodecahedron. Similar "towers" through the other ten faces lead us to identify all opposite pairs of faces.

In order to prove that the fundamental domain of I^* is indeed the dodecahedron requires an analysis of how $SU(2)$ acts on S^3. We assume that $SU(2)$ acts on C^2 on the right, that is, if $u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a\bar{a} + b\bar{b} = 1$, then $u(z,w) = (z,w)\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$. Identify

$SU(2)$ with the unit 3-sphere in C^2 by taking $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ to (a,b). Thus S^3 acts on itself; we want a simple geometric picture of this action.

The complex lines in C^2 intersect only at the origin, so the complex lines intersected with S^3 give a decomposition (foliation) of S^3 into circles. (These circles are also the orbits under the circle action $\lambda(z,w) = (\lambda z, \lambda w)$, $\lambda \in S^1 \subset C$.) By sterographic projection identify S^3 with $R^3 \cup \infty$, with coordinates (r,s,t) on R^3. Assume the complex line $w = 0$ (the z-axis) intersects S^3 in $(r\text{-axis} \cup \infty) = S_1$ and the line $z = 0$ intersects S^3 in the unit circle S^3 in the (s,t) plane; in particular $(1,0,0,0)$ in S^3 goes to $(0,0,0)$ in R^3. The other complex lines intersect S^3 in the following kinds of circles; the complement of $S_1 \cup S_2$ in S^3 is a union of "concentric" tori (since S^3 is the join of S_1 and S_2). Each torus is the union of disjoint circles obtained from the -45° lines in the square by identifying opposite sides of the square in the orientation preserving way. In particular, we orient these circles continuously so that S_2 is oriented consistently with the usual orientation of the (st)-plane and S_1 has the same orientation as the z-axis. (If we think of S^2 as the unit disk in the (st)-plane with S_2 collapsed to a point, then each circle intersects S^2 exactly once; this defines the Hopf map $H : S^3 \to S^2$.)

Armed with this picture, the action of $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \in SU(2)$. $\lambda \in S^1 \subset C$, is easy to see. It twists the circle S_1 by an angle λ, and the circle S_2 by the angle $\bar{\lambda}$; the orbits of the induced action on the tori are perpendicular to those of λ.

There is, for each circle S in S^3 and point p in S, a copy of R^3 in R^4 perpendicular to S at p. The intersection with S^3 of this perpendicular R^3 will be called the perpendicular sphere at p in S. Clearly $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$ carries the perpendicular sphere at p in S_1 to the perpendicular sphere at λp.

In general, $n = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ can be described similarly. It is the product of rotations through θ in the *real* plane L_u spanned by

$(1,0)$ and (a,b) and its orthogonal complement L'_u. This can be seen as follows. If $a = \alpha + i\beta$ and $b = \gamma + i\delta$ then the embedding $i : SU(2) \to SO(4)$ gives

$$i(u) = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & -\gamma & \beta & \alpha \end{pmatrix}$$

with $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.
With respect to the basis

$$(1,\ 0,\ 0,\ 0),\quad \left(0,\ \frac{\beta}{\mu},\ 0,\ \frac{-\nu}{\mu}\right),$$

$$\left(0,\ \frac{\gamma}{\mu},\ \frac{\delta}{\omega},\ \frac{\beta\gamma}{\mu\nu}\right),\quad \left(0,\ \frac{\delta}{\mu},\ \frac{-\gamma}{\nu},\ \frac{\beta\delta}{\mu\nu}\right),$$

where $\mu = \sqrt{1-\alpha^2}$ and $\nu = \sqrt{\gamma^2+\delta^2}$,

$$i(u) \cdot \begin{pmatrix} \alpha & \sqrt{1-{}^{2}} & 0 & 0 \\ -\sqrt{1-\alpha^2} & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\sqrt{1-\alpha^2} \\ 0 & 0 & \sqrt{1-\alpha^2} & \alpha \end{pmatrix}$$

and $\cos\theta = \alpha = \mathrm{Re}(a)$. Thus u (or any other element of $L_u \cap S^3$) defines a Hopf-like decomposition of S^3 into circles.

Let S_u be that circle in S^3 which in R^3 is the line $t \cdot \left(\frac{\beta}{\mu}, \frac{\gamma}{\mu}, \frac{\delta}{\mu}\right)$, $t \in R$. Here $\left(\frac{\beta}{\mu}, \frac{\gamma}{\mu}, \frac{\delta}{\mu}\right)$ is the image of $(1,0,0)$ under the change of basis, so S_u is the image of S_1. Accordingly, u carries spheres perpendicular to S_u to other such spheres.

We now embed I^* in $SU(2)$ as described above. Place the dodecahedron with center at $0 \in R^3$ so that the barycenter of a face is tangent to $S^2 \cong C \cup \infty$ at the point $(1,0,0)$ in R^3, which corresponds to the point 0 in C under stereographic projection. Twist the dodecahedron by $2\pi/5$; this element g_o corresponds in C to multiplication by λ^2, where $\lambda = e^{\pi i/5}$, or, equivalently, to the linear fractional transformation whose matrix is $\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}$. Thus g_o is covered by $\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}$ in $SU(2)$. We have examined the action of such an element on S^3; it maps the sphere perpendicular to $e^{-\pi i/10} \in S_1$ to that perpendicular to $e^{\pi i/10}$. Thus the Z_{10} subgroup of $SU(2)$ covering the rotations of the dodecahedron about $(1,0,0)$ has a fundamental domain the region lying between these two perpendicular spheres--a lens shaped region (which gives its name to the Lens space S^3/Z_{10}).

Now let $z = u + iv$ be the barycenter of another face of the dodecahedron tangent to $S^2 = C \cup \infty$. Let $\rho = \sqrt{1 + z\bar{z}} = \sqrt{1 + (u^2 + v^2)}$ and note that the linear fractional transformation carrying 0 to z is given by the matrix

$$A = \begin{pmatrix} \frac{1}{\rho} & \frac{-\bar{z}}{\rho} \\ \frac{z}{\rho} & \frac{1}{\rho} \end{pmatrix} .$$

The $2\pi/5$ twist about this barycenter, denoted g_z, then corresponds to a matrix $A\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}A^{-1}$ in $SU(2)$. Let $\lambda = e + if$, and denote, as above, the entries in the matrix image of $A\begin{pmatrix}\lambda & 0\\ 0 & \bar{\lambda}\end{pmatrix}A^{-1}$ in $SO(4)$ by $\alpha, \beta, \gamma, \delta$. An easy calculation shows that $\alpha = e$, $\beta = f/\rho^2(2 - \rho^2)$, $\gamma = \frac{2fv}{\rho^2}$, $\delta = \frac{2fu}{\rho^2}$.

As before, let S_z be a circle in S^3 such that $S_z \cap R^3 = t(\beta/\mu, \gamma/\mu, \delta/\mu)$. Then g_z carries spheres perpendicular to S_z to other such spheres. In this case $\mu = \sqrt{1 - e^2} = f$

$$\left(\frac{\beta}{\mu}, \frac{\gamma}{\mu}, \frac{\delta}{\mu}\right) = \left(\frac{2-\rho^2}{\rho^2}, \frac{2v}{\rho^2}, \frac{2\mu}{\rho^2}\right) .$$

But the coordinates of z in R^3 under stereographic projection are

$$\left(\frac{2-\rho^2}{\rho^2}, \frac{2u}{\rho^2}, \frac{2v}{\rho^2}\right) .$$

Comparing the two vectors in R^3 we deduce (after the orthogonal rotation which switches the last two coordinates) that rotation of the dodecahedron about the barycenter at z lifts in $SU(2)$ to the same action as rotation about the barycenter at 0, except the axis in R^3 of the translation of S^3 *now points through* z *instead of through* $(1,0,0)$. Thus the axis of the (lens-shaped) fundamental domain of g_z passes through z. The intersection of all the fundamental domains of all rotations about barycenters of faces is then the intersection of those lenses whose axes point in the direction of the barycenters. But this intersection is precisely the fundamental domain of I^*, since it is easy to see that all elements of I are compositions of rotations about barycenters of faces. But the intersection of these lenses is clearly the dodecahedron. Furthermore, we have seen that the action of I^* on S^3 identifies opposite sides of the lenses (hence of the dodecahedron) with a $\pi/5$ twist. This completes the proof.

Incidentally, it is possible to explicitly calculate generators and relations for I^* by using description 5) for ∂P, as in [18]. This is then a roundabout proof that $I^* = \{(x,y) \mid x^3 = y^5 = (xy)^2\}$.

Equivalence of descriptions (3) and (4). The proof is a medley of [12] and [10]. Our aim is to find a homeomorphism $P : C^2/I^* \to f^{-1}(0)$ where $f : C^3 \to C$ is $f(z_1,z_2,z_3) = z_1^2 + z_2^3 + z_3^5$. We will find three homogeneous polynomials $p_1, p_2, p_3 : C^2 \to C^1$, and define $\overline{P} = (p_1,p_2,p_3) : C^2 \to C^3$. We must show that

(i) $\overline{P}$ is invariant under action by I^* so that $\overline{P}$ defines $P : C^2/I^* \to C^3$.

(ii) $p_1^2 + p_2^3 + p_3^5 = 0$ so that image $(P) \subset f^{-1}(0)$.

(iii) $d\overline{P}$ has rank 2 on $C^2 - 0$ and thus $P\ (C^2/I^*)-0$ is a covering map.

(iv) $\overline{P}^{\,-1}(\text{point}) = 120$ points, so P is one-to-one and a homeomorphism.

$$\text{10 elements} \qquad \pm \begin{pmatrix} \varepsilon^{3\mu} & 0 \\ 0 & \varepsilon^{2\mu} \end{pmatrix}$$

$$\text{10 elements} \qquad \pm \begin{pmatrix} 0 & -\varepsilon^{2\mu} \\ \varepsilon^{3\mu} & 0 \end{pmatrix}$$

$$\text{50 elements: } \pm \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon^{3(\mu+\omega)}(\varepsilon-\varepsilon^{4}) & \varepsilon^{3(\omega-\mu)}(\varepsilon^{2}-\varepsilon^{3}) \\ \varepsilon^{3(\mu-\omega)}(\varepsilon^{2}-\varepsilon^{3}) & \varepsilon^{-3(\mu+\omega)}(\varepsilon-\varepsilon^{4}) \end{pmatrix}$$

$$\text{50 elements: } \pm \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon^{3(\mu-\omega)}(\varepsilon^{2}-\varepsilon^{3}) & \varepsilon^{-3(\mu+\omega)}(\varepsilon-\varepsilon^{4}) \\ -\varepsilon^{3(\mu+\omega)}(\varepsilon-\varepsilon^{4}) & -\varepsilon^{3(\omega-\mu)}(\varepsilon^{2}-\varepsilon^{3}) \end{pmatrix}$$

This yields the polynomials

$$p_3 = -(1728)^{1/5}\, z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10})$$

$$p_1 = (z_1^{30} + z_2^{30}) + 522(z_1^{25}\, z_2^{5} - z_1^{5}\, z_2^{25}) - 10005(z_1^{30} + z_1^{10}\, z_2^{20})$$

$$p_2 = -(z_1^{20} + z_2^{20}) + 228(z_1^{15}\, z_2^{5} - z_1^{5}\, z_2^{15}) - 494 z_1^{10}\, z_2^{10}\ .$$

The reader may verify directly that $p_1^2 + p_2^3 + p_3^5 = 0$, but the following argument is both more elegant and requires no calculation.

Consider the complex vector space V of homogeneous polynomials of degree 60; V has dimension 61, with basis $z_1^{i}\, z_2^{60-i}$, $i = 0,\dots,60$. There is a 2-dimensional subspace $W = \{\lambda p_3^5 + \mu p_1^2\}$, for $\lambda,\mu \in C$. Given a barycenter (a,b) of a face of the icosahedron, the annihilator A of the 1-dimensional subspace $az_2-bz_1 = 0$ has dimension 60. Thus $\dim(W \cap A) \geq 1$, so for some fixed λ and μ, not both zero, $(\lambda p_3^5 + \mu p_1^2)(z_1, z_2) = 0$ if $az_2 - bz_1 = 0$.

The orbits of points in CP^1 under the action of I are of four types:

(1) the 12 vertices of the icosahedron

(2) the 30 barycenters of edges

(3) the 20 barycenters of faces

(4) orbits containing 60 points.

Since p_1, p_2, and p_3 are invariant under I^*, it follows that the zeroes of $\lambda p_3^5 + \mu p_1^2$ must consist of complex lines through entire orbits. Suppose there are ω_i orbits of type i, $i = 1,2,3,4$, in the zeroes of $\lambda p_3^5 + \mu p_1^2$, multiplicities included. Then degree $(\lambda p_3^5 + \mu p_1^2) = 60 = 12\omega_1 + 30\omega_2 + 20\omega_3 + 60\omega_4$. Since $\lambda p_3^5 + \mu p_1^2$ is zero on the complex line through a barycenter of a face, it follows that $\omega_3 \neq 0$, but then $\omega_3 = 3$ and $\omega_1 = \omega_2 = \omega_4 = 0$. Thus $\lambda p_3^5 + \mu p_1^2$ has the same zeroes as p_2^3, so $\lambda p_3^5 + \mu p_1^2 = \nu p_2^3$. We redefine p_3 to be the old p_3 divided by $\lambda^{1/5}$ and so on, so that $p_1^2 + p_2^3 + p_3^5 = 0$. We have now satisfied properties (i) and (ii).

To show that $\overline{P} : C^2 \to f^{-1}(0)$ or $P : C^2/I^* \to f^{-1}(0)$ is locally bihilomorphic off zero, it suffices to prove that the matrix

$$d\overline{P} = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} & \frac{\partial p_2}{\partial z_1} & \frac{\partial p_3}{\partial z_1} \\ \\ \frac{\partial p_1}{\partial z_2} & \frac{\partial p_2}{\partial z_2} & \frac{\partial p_3}{\partial z_2} \end{pmatrix}$$

has rank 2 everywhere. Note that

point, which we call $Q(x)$. Q is clearly smooth and one-to-one; it is onto since P is.

Equivalence of descriptions (3) and (6). We will define a circle action on the link L of the singularity $z_1^2 + z_2^3 + z_3^5 = 0$, an action which gives L the required Seifert manifold structure. For $\gamma \in S^1 \subset C$, let $\gamma(z_1,z_2,z_3) = (\gamma^{15} z_1, \gamma^{10} z_2, \gamma^6 z_3)$. Clearly this circle action on C^3 leaves L invariant.

The orbits of S^1 are principal if all $z_i \neq 0$, for if $\gamma(z_1,z_2,z_3) = (z_1,z_2,z_3)$ then $\gamma^6 = \gamma^{10} = \gamma^{15} = 1$, so $\gamma = 1$. The exceptional orbits are the three orbits $z_1 = 0$, $z_2 = 0$, $z_3 = 0$.

S^1 acts on the orbit $z_3 = 0$ by $\gamma(z_1,z_2,0) = (\gamma^{15}, \gamma^{10} z_2, 0)$ so $Z_5 \subset S^1$ acts trivially on the orbit. Furthermore if $\omega \in S^1$ satisfies $\omega^5 = 1$, then ω acts on a disk perpendicular to the orbit via complex multiplication by $(\omega)^6 = \omega$. Thus the orbit is exceptional of type $(5,1)$. Similarly $z_2 = 0$, $z_1 = 0$ are exceptional orbits of type $(3,1)$ and $(2,1)$.

Next we show that any Seifert manifold M whose only exceptional orbits are of type $(2,1)$, $(3,1)$ and $(5,1)$ and which is an integral homology sphere is a Seifert manifold with quotient space S^2 and cross-section obstruction -1. This will complete the proof, for we know from the equivalence of 3 and 5 that $\pi_1(L) = \{x,y: x^3 = y^5 = (yx)^2\}$ so $H_1(L) = 0$.

First note that if the quotient space $\overline{M}$ of M by the circle action is of genus g, then the first homology of the (trivial) circle bundle obtained by deleting the exceptional orbits has rank $2g + 3$. Sewing back the 3 exceptional orbits can at most decrease the rank to $2g$, because sewing in a copy of $S^1 \times B^2$ adds only the relation corresponding to $s \times \partial B^2$. Thus $g = 0$ and $\overline{M} = S^2$.

To calculate the cross-section obstruction b, construct a presentation for $\pi_1(M)$ as follows. A cross-section for the circle bundle away from the exceptional orbits is a 3-punctured sphere with fundamental group $\{q_1,q_2,q_3;\ q_1q_2q_3 = 1\}$, where each q_i is a path around an exceptional orbit. Let $L \in \pi_1(M)$ be represented by a principal orbit.

Let (j,k) denote the path in $S^1 \times \partial B^2$ which goes j times around S^1 and k times around ∂B^2. Attach the exceptional orbits by d_i, $i = 2,3,5$. We may assume that $d_2(-c,1) = q_1$, $d_3(0,1) = q_2$, $d_5(0,1) = q_3$. But $d_i(1,i) = (0,1)$ which is null-homotopic in $S^1 \times B^2$. Thus adding the exceptional orbits introduces the relations $d_i(1,i) = 0$. Thus $q_1^2 = d_2(-2c,2) = d_2(-2c-1,0) = h^{-2c-1}$, $q_2^3 = d_3(0,3) = d_3(-1,0) = h^{-1}$, $q_3^5 = d_5(0,5) = d_5(-1,0) = h^{-1}$. Thus $\pi_1(M) = \{q_1,q_2,q_3,h; q_1^2\, h^{2c+1} = q_2^3\, h = q_3^5\, h = [q_i,h] = q_1q_2q_3 = 1\}$. Eliminate q_1 by $q_1 = (q_2q_3)^{-1}$ and h by $h = q_2^{-3}$ and abelianize to obtain

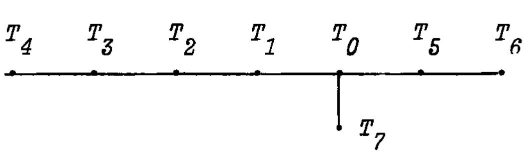
$$H_1(M) = \{q_2,q_3; 2q_3 + (6c+5)q_2 = 5q_3 - 3q_2 = 0\}.$$

Thus $H_1(M)$ is of order

$$\det \begin{vmatrix} 6c+5 & 2 \\ -3 & 5 \end{vmatrix} = 30c + 25 + 6.$$

Since $H_1(M) = 0$, $c = -1$.

Equivalence of descriptions (6) and (1). Examine the structure of the plumbing construction P^4, 8 copies of the cotangent disk bundle T plumbed together as shown:



This may be viewed as plumbing 3 "arms" of length 4, 2, and 1, to the central T_0. That part of ∂P^4 lying in each arm has a very simple description. In particular, $T_1 \cap \partial P^4$ is an S^1 fiber bundle over S^2 with neighborhoods of two fibers removed. But removing two fibers leaves a trivial bundle, so $T_1 \cap \partial P^4 \cong S^1 \times S^1 \times I$ for $i = 1,2,3,5$, and $T_j \cap \partial P^4 = S^1 \times B^2$ for $j = 4,6,7$. Thus each arm, consisting, for example of $T_1 \cup T_2 \cup T_3 \cup T_4$, intersects in a copy of $S^1 \times B^2$. Hence ∂P^4 is obtained from ∂T_0 by removing three tubular neighborhoods $(S^1 \times B^2)_i$ of fibers in ∂T_0 and, to each $(S^1 \times \partial B^2)$ boundary component, attaching a copy of $S^1 \times B^2$ by some (linear) attaching map $g_i : (S^1 \times \partial B^2)_i \to S^1 \times \partial B^2$.

There is a natural circle action on ∂T_0, in which the circle acts on each fiber by rotation. Remove $(S^1 \times B^2)_i$, $i = 1,2,3$ and attach three copies of $S^1 \times \partial B^2$ by $\{g_i\}$. The action on $(S^1 \times B^2)_i$ extends linearly over each attached $S^1 \times B^2$, so the circle action extends over ∂P^4. We will verify that this circle action gives the required Seifert manifold structure by calculating g_i.

LEMMA 1. The attaching map $g_i : (S^1 \times \partial B^2)_i \to S^1 \times \partial B^2$ for an arm of length $m \geq 0$ is given by the matrix

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^m = \begin{pmatrix} m+1 & m \\ -m & 1-m \end{pmatrix} .$$

Proof. This is certainly the case for $m = 0$, that is, for ∂T_0 itself. The proof is by induction. Suppose it is true for an arm of length $m \geq 0$.

Let T_m denote the copy of T at the end of the arm to which we plumb T_{m+1}. Since T_{m+1} has Euler class -2, it is made from two charts, $(B^2 \times B^2)^1$ and $(B^2 \times B^2)^2$, and we identify $(B^2 \times \partial B^2)^1$ with $(B^2 \times \partial B^2)^2$ by extending the map $(\partial B^2 \times \partial B^2)^1 \to (\partial B^2 \times \partial B^2)^2$ given by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ linearly across $(B^2 \times \partial B^2)^1$. Plumb T_{m+1} in along $(B^2 \times B^2)^2$. Then a copy $(S^1 \times B^2)^3$ in ∂T_m is identified with $(B^2 \times \partial B^2)^1$ by switching factors, so the attaching map $(S^1 \times \partial B^2)^3 \to (\partial B^2 \times \partial B^2)^1$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. But by induction hypothesis, the map $(S^1 \times \partial B^2)_i \to (S^1 \times \partial B^2)^3$ is given by $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^m$. Thus the attaching map $(S^1 \times \partial B^2)_i \to (S^1 \times \partial B^2)^2$ is given by

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^m = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{m+1},$$

proving the lemma.

LEMMA 2. *Each arm of length m attached to T_0 adds an exceptional orbit of type $(m+1, 1)$ and decreases the cross-section obstruction by one.*

Proof. The proof is based on the matrix identity

$$\begin{pmatrix} m+1 & m \\ -m & 1-m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} m+1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Thus from Lemma 1 g_i is the composition of three automorphisms $(S^1 \times \partial B^2) \circlearrowleft$. The first, represented by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has no effect on the circle action, but changes the cross-section $(0,1)$ to $(1,1)$. The second automorphism is just d_{m+1}. The original cross-section obstruction c is the obstruction to extending the cross-section $d_{m+1}(1,1)$. Then the obstruction of extending $d_{m+1}(0,1)$ is $c - 1$. Thus the composition

$$\begin{pmatrix} m+1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

represents the addition of an exceptional orbit of type $(m+1, 1)$, and a decrease by 1 in the cross-section obstruction. The third automorphism, represented by $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ extends to an automorphism $S^1 \times B^2 \circlearrowleft$ and thus has no effect on the homeomorphism type. This proves the lemma.

Since ∂P^4 is obtained from T_Q, which has cross-section obstruction 2 (Euler class -2), by attaching arms of length 1, 2, and 4, it follows that the cross-section obstruction is -1, and ∂P^4 has exceptional orbits of type $(2,1)$, $(3,1)$ and $(5,1)$.

Equivalence of descriptions (1) and (2). The equivalence of descriptions (1) and (2) follows from the definitions; see definition of (2) above.

Equivalence of descriptions (3) and (8). We sketch the proof of Milnor [12]. The torus knot of type $(2,3)$ is the knot which wraps around the standard torus in R^3, twice in one direction and three times in the other. In other words it is the image in $S^3 \subset C^2$ of the circle $S^1 \subset C$ under the map $t \to \frac{1}{\sqrt{2}}(t^3, t^2)$. This is the intersection of S^3 and the variety $\{(z_1, z_2) \in C^2 \mid z_1^2 = z_2^3\}$.

Let L be the link of the singularity $z_1^2 + z_2^3 + z_3^5 = 0$, and M be the 5-fold branched cover of the trefoil knot.

Evidently $V = \{(z_1, z_2, z_3) \in C^3 - 0 \mid z_1^2 + z_2^3 + z_3^5 = 0\}$ is the 5-fold branched cover of $C^2 - 0$ along $B = \{(z_1, z_2) \in C^2 - 0 \mid z_1^2 + z_2^3 = 0\}$. Indeed the projection $(z_1, z_2, z_3) \to (z_1, z_2)$ is a 5-fold cover away from $z_1^2 + z_2^3 = 0$ corresponding to the 5 roots of $z^5 \neq 0$, but is a homeomorphism when $z_1^2 + z_2^3 = 0 = z_3^5$. R^+ acts on V and $C^2 - 0$ by $t(z_1, z_2, z_3) = (t^{1/2} z_1,\ t^{1/3} z_2,\ t^{1/5} z_3)$ and $t(z_1, z_2) = (t^{1/2} z_1,\ t^{1/3} z_2)$. The action commutes with projection. Since each orbit of R_+ intersect L precisely once, $V/R_+ \cong L$. Similarly $C^2 - 0/R_+ \cong S^3$ and the induced map $L \to S^3$ is a branched 5-fold cover over $B/R_+ \cong$ trefoil knot.

Equivalence of descriptions (2) and (7). We need to show that doing surgery on the framed link Λ of (2) gives the same 3-manifold as doing surgery on the left handed trefoil knot using the *-1* framing.

The first author shows that two framed links Λ and Λ' yield the same 3-manifold if and only if they are related by a series of link operations of two kinds [9]:

$\mathcal{O}_1$: Add to or subtract from a link an unknotted circle with framing ± 1, which is separated from the other circles by an embedding S^2 in S^3.

$\mathcal{O}_2$: Given two components γ_0 and γ_1 of an oriented, framed link, push γ_1 off itself, using its given framing, to obtain γ_1'. Join γ_0 and γ_1 by a strip $b : I \times I \hookrightarrow S^3$ such that $b(I \times I) \cap \gamma_i = b(i \times I)$, $i = 0,1$. Then substitute for γ_0 and γ_1 the circles γ_1' and $\gamma_0 \#_b \gamma_1 = \gamma_0 \cup \gamma_1 \cup b(I \times \partial I) - b(\partial I \times I)$.

The framing for γ_1' is the same as that for γ_1; that for $\gamma_0 \#_b \gamma_1$ is the sum of the framings of γ_0 and γ_1 plus or minus twice the linking number of γ_1 with γ_0. The sign is plus if and only if $b(I \times I)$ can be oriented consistently with γ_0 and γ_1.

The full strength of this theorem is unnecessary. Here we use only the "easy" part, that if two links are related by $\mathcal{O}_1$ and $\mathcal{O}_2$, then the corresponding 3-manifold are homeomorphic. Indeed, $\mathcal{O}_1$ corresponds to taking connected sum with or splitting off a copy of the complex projective plane $\pm CP^2$, with one of its orientations, from the trace of the surgery. This follows immediately from the fact that $\pm CP^2$ - *(4-disk)* is the Hopf disk bundle over S^2 with Euler class ± 1.

$\mathcal{O}_2$ corresponds to sliding the 2-handle attached along γ_0 (in the trace of the surgery) across the 2-handle along γ_1.

LEMMA 3. *If we change a portion of a framed link as in Figure 5 below, then the 3-manifold resulting from surgery is not changed.*

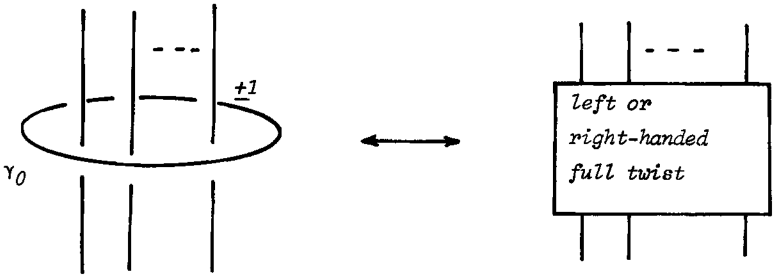


Fig. 5.

If γ has framing n in the left case, then it will have framing $n \pm (\ell(\gamma_0, \gamma))^2$ in the right case.

The proof, given in [9], is a straightforward application of $\mathcal{O}_1$ and $\mathcal{O}_2$, and can be worked out easily by the reader for one or two strands through γ_0.

By a series of applications of Lemma 3 we change the framed link Λ to the *-1* trefoil knot. First we introduce three unknots with *+1* framing $(\mathcal{O}_1)$ and then slide the end circles of Λ over them $(\mathcal{O}_2)$ to get

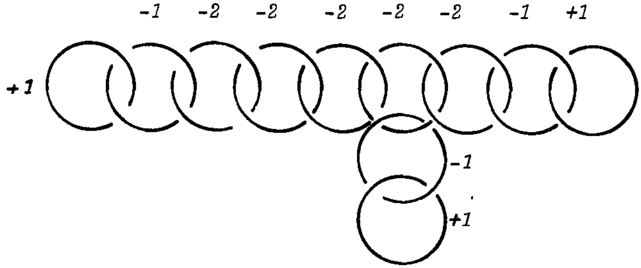


Fig. 6.

Then we remove the *-1* circles, using the lemma until we get

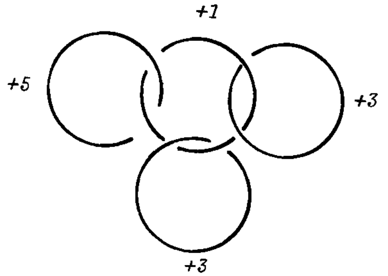


Fig. 7.

Removing the *+1* circle, we get

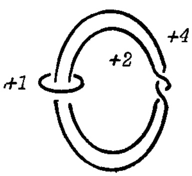


Fig. 8.

Blowing down the *+1* circle gives

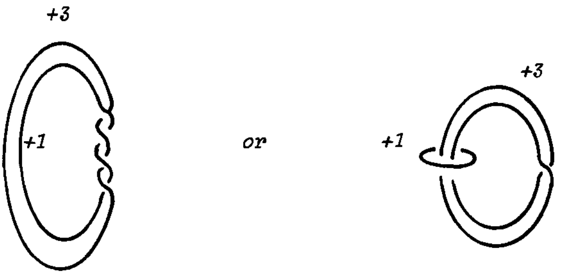


Fig. 9.

And one last application of the lemma finishes the proof.

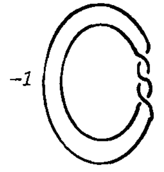


Fig. 10.

Equivalence of descriptions (7) and (8): We will find a handle body or surgery description of the 5-fold branched cover of the right handed trefoil knot K. First we perform a surgery on an unknot J with framing 1, so that S^3 is the result. But if the unknot is chosen appropriately, K is unknotted in the new S^3 (see Figure 11). It is easy to see that the 5-fold branched cover is still S^3, but the curve J lifts to 5 copies, $J_1,\ldots,J_5$, whose framings can be calculated from the formula

$$\ell\left(\sum_{i=1}^{5} J_i, \sum_{i=1}^{5} J_i\right) = \sum_{i,j=1}^{5} \ell(J_i,J_j) = 5\ (J,J) = 5$$

which implies by symmetry that

$$\sum_{i=1}^{5} \ell(J_1,J_i) = 1.$$

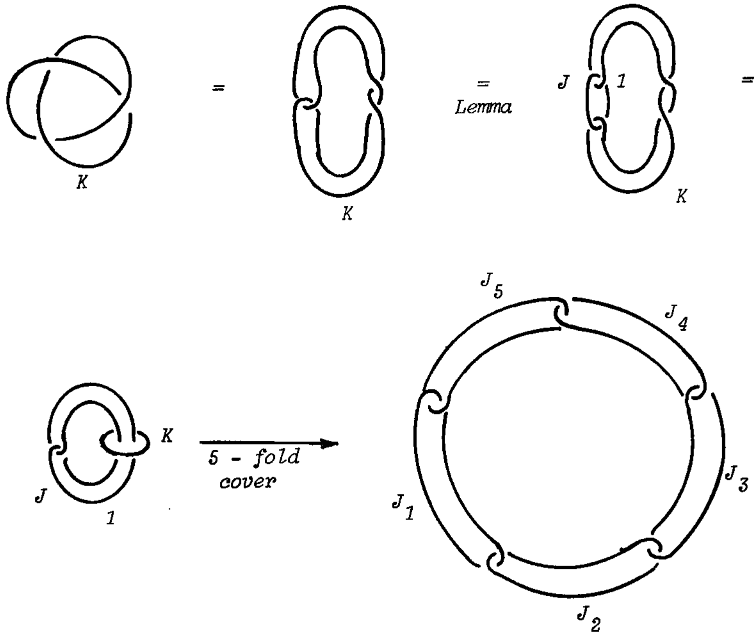


Fig. 11.

We see by inspection that $\ell(J_1, J_2) = \ell(J_1, J_5) = 1$ and $\ell(J_1, J_3) = \ell(J_1, J_4) = 0$ so that $\ell(J_1, J_1) = -1$ and hence $\ell(J_i, J_i) = -1$. The point now is that if surgery on J in S^3 gives K in S^3, then surgery on $J_1, \ldots, J_5$ gives the 5-fold branched cover of K. To see this surgery, we apply Lemma 3 several times, first to say, J_1 and J_4, obtaining

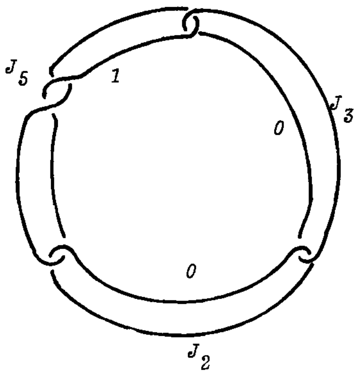


Fig. 12.

then to J_5,

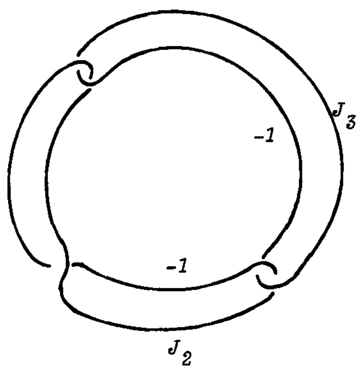


Fig. 13

and finally to, say, J_3.

Fig. 14.

Equivalence of descriptions (6) and (2) or (7). We have seen in description (6) that the Seifert surface is equivalent to surgery on 3 anti-Hopf circles (Figure 3) with framings 1, 2, and 4. But exactly this framed link turns up while showing that descriptions (2) and (7) are equivalent.

III. OTHER POSSIBILITIES

The equivalences proven here are not necessarily the most direct paths between two points. The interested reader would find it quite rewarding to construct shortcuts. Here are some which exist in the literature.

The equivalence of (1) and (3) can be seen by resolving the singularity $z_1^2 + z_2^3 + z_3^5 = 0$. In fact, the minimal resolution of $z_1^2 + z_2^3 + z_3^5 = 0$ is a complex manifold homeomorphic to P^4. The resolution of the singularity is explicitly done in [11, p. 23-27]. Perhaps a more piquant approach, through, is to use the more general theorems on resolution of singularities found, for example in [6], [7], [1], [15], to discover the connection between the resolution of those singularities corresponding to the platonic solids (e.g., the dodecahedron) and the Dynkin diagrams used to classify semi-simple Lie groups.

In [16] the second author sketches a proof of $(6) \Longleftrightarrow (7)$ by studying the circle action on S^3 given by $\gamma(z_1, z_2) = (\gamma^3 z_1, \gamma^2 z_2)$. The trefoil knot $z_1^2 + z_2^3 = 0$ is an orbit of this action.

A Heegard splitting for ∂P^4 is drawn on page 19 of [5] and on page 245 of [14]. The latter shows that his Heegard splitting coincides with our description (7). The reader can construct his own Heegard splitting via, say, the 2-fold branched cover of the $(3,5)$-torus knot K. Note that we can decompose S^3 into two $B^2 \times I$'s such that for each one, we have $(B^2 \times I) \cap (S^3,K) \cong (B^2 \times I, ((-\frac{1}{2},0) \times I) \cup ((0,0) \times I) \cup ((\frac{1}{2},0) \times I))$; the 2-fold branched cover of $B^2 \times I$ over 3 unknotted strands is the solid 2-holed torus, i.e. $B^3 \cup$ (two 1-handles). What remains is to "see" the homeomorphism by which the two are glued together.

Description (8) for ∂P^4 can be extended to give a definition of P^4 as a p-fold cover of B^4 branched over a certain Seifert surface of the (q,r)-torus knot, $\{p,q,r\} = \{2,3,5\}$. The surface is obtained by pushing into B^4 the fiber of the map $S^3 - K \to S^1$ given by $(z,w) \to (z^q+w^r)/|z^1+w^r|$, $(z,w) \in C^2$. A Seifert surface for the $(3,5)$-torus knot is drawn in Figure 15. Its double branched cover is exactly Figure 2.

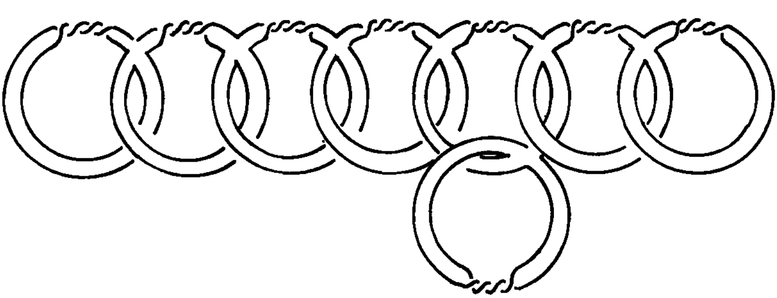


Fig. 15.

If we take the usual Seifert surface for the right trefoil knot, Figure 16, push it into B^4 and take the 5-fold branched cover, we get Figure 17. S. Akbulut and J. Harer pointed out this description; it occurs naturally as a complex submanifold of the Kummer surface. It is not hard to slide 2-handles over 2-handles to get from Figure 17 to Figure 2. We leave a description of the 3-fold cover to the reader.

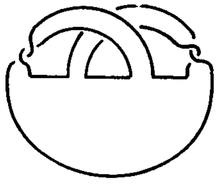


Fig. 16.

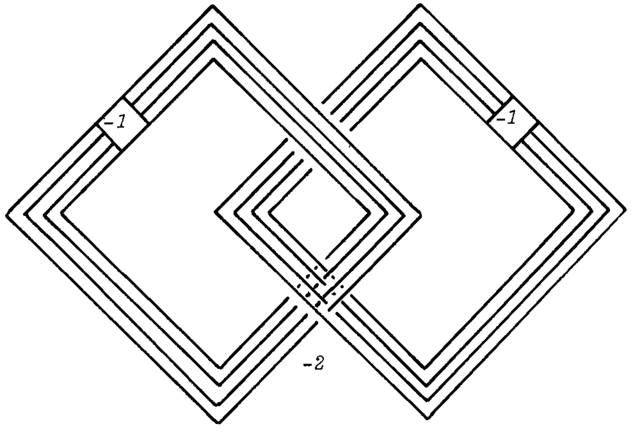


Fig. 17. The -1 means one full left-handed twist. Each circle has framing -2.

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