Mock exam Geometry 2022 with solutions

1. Suppose $n \geq 2$ is a positive integer and imagine an n-simplex $J = [v_0, \dots, v_n]$ in $\mathbb{R}^n$. K is the simplicial complex in $\mathbb{R}^n$ consisting of all the faces of J that have dimension 2 or less.

   (a) Write down the Euler characteristic of K as a function of n.
   For a simplex $[S]$ all k-element subsets of $T \subset S$ correspond to a face $[T]$ of the simplex and all faces are like that. Therefore $\#K_0 = \#J_0 = n + 1$ and $\#K_1 = \#J_1 = \binom{n+1}{2}$ and $\#K_2 = \#J_2 = \binom{n+1}{3}$. Since K contains no simplices of dimension > 2 we can compute the Euler characteristic as $\chi(K) =$

   $$\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} = n+1-\frac{1}{2}(n+1)n+\frac{1}{6}n(n+1)(n-1) = \frac{n+1}{6}(n^2-4n+6)$$

   (b) For any $i \in \{0, 1, \dots, n\}$, denote by $R_i : \mathbb{R}^n \to \mathbb{R}^n$ the affine reflection in the affine hyperplane spanned by the $(n-1)$-dimensional face of J that does NOT contain v_i. Prove that for any $i, j \in \{0, 1, \dots, n\}$ the composition $R_i \circ R_j$ is an affine rotation.
   The composition of two affine reflections in affine planes X and Y is an affine rotation whenever X and Y intersect. In this case $X = [\{v_0, \dots, v_n\} \setminus \{v_i\}]$ and $Y = [\{v_0, \dots, v_n\} \setminus \{v_j\}]$ so that $X \cap Y = [\{v_0, \dots, v_n\} \setminus \{v_i, v_j\}]$. This is non-empty because $n \geq 2$.

   (c) Define $L = K \cup \{R_0(\sigma) | \sigma \in K\}$. Prove that L is a simplicial complex.
   For any simplex $[S]$ and any affine reflection R we have $R([S]) = [R(S)]$ so the reflection of a simplex is again a simplex. Now we should check that the intersection of two simplices in L is again in L. The reflection hyperplane contains all vertices $v_1, \dots v_n$ but not v_0. This means that$R_0(v_i) = v_i$ whenever $i > 0$ and $R_0(v_0) \neq v_0$. The simplices in L are of the form $[S]$ where $\#S \leq 3$ and $S \subset \{R_0(v_0), v_0, v_1, \dots v_n\}$ and S cannot contain both v_0 and $R_0(v_0)$. The intersection of two simplices in L are therefore of three types: either we have two simplices in K that intersect and there we already know that they intersect in a common face inside K because J is a simplicial complex. Applying R_0 which is its own inverse we come to the same conclusion when both simplices that we want to intersect are in $R_0(K)$. Finally if one simplex $[T]$ is in K and the other $[S]$ is not then their intersection cannot contain v_0 and also not $R_0(v_0)$. This implies that $[T] \cap [S] = [T \setminus \{R_0(v_0)\}] \cap [S]$ as this intersection takes place in K we are done.

   (d) Find an explicit example of a simplex J as above such that $|L|$ is not a convex polyhedron in $\mathbb{R}^n$.
   We can choose $J = [e_1 - e_2, 0, e_2]$ then $R_0(x, y) = (-x, y)$ is the reflection in the y-axis. L contains the simplices $\{e_1 - e_2\}$ and $\{-e_1 - e_2\}$ so if $|L|$ were to be a convex polyhedron then $|L|$ should also contain the point $\frac{1}{2}(e_1 - e_2 - e_1 - e_2) = -e_2$. However $|L| = [e_1 - e_2, 0, e_2] \cup [-e_1 - e_2, 0, e_2]$ does not contain $-e_2$.

2. Define $f(x, y) = x^2 - y - 1$. Homogeneous coordinates in $\mathbb{P}^2$ are taken with respect to the standard basis of $\mathbb{R}^3$. Polarity is taken with respect to the standard inner product on $\mathbb{R}^3$.

   (a) Find a non-zero polynomial in three variables $F(x, y, z)$ such that $P(X(f) \times \{1\}) \subset P(X(F)) \subset \mathbb{P}^2$.
   We homogenize f to get $F(x, y, z) = x^2 - yz - z^2$. Since $F(x, y, 1) = f(x, y)$ we have $X(f) \times \{1\} \subset X(F)$ and applying P on both sides finishes the proof.

   (b) Give the homogeneous coordinates of a point in $P(X(F)) \setminus P(X(f) \times \{1\})$.
   Since $F(x, y, 0) = x^2$ we could pick the point $[0 : 1 : 0]$ for example.

(c) Compute the polar of the projective line through the points $[1:1:1]$ and $[1:2:1]$ in $\mathbb{P}^2$. If we set $v = (1,1,1)$ and $w = (1,2,1)$ then the pline through $\underline{v}$ and $\underline{w}$ is $P(U)$ where $U = \underline{\{v,w\}}$. To find the polar we first determine $U^{\perp}$. To find $U^{\perp}$ we have to find which vectors $a = (a_1, a_2, a_3)$ are perpendicular to both v and w. So we should have $a_1 + a_2 + a_3 = 0 = a_1 + 2a_2 + a_3$ meaning $a_2 = 0$ and $a_3 = -a_1$ in other words $U^perp = \underline{(1,0,-1)}$ and the polar to the pline $P(U)$ is the set $P(U^{\perp}) = \{\underline{(1,0,-1)}\}$.

(d) Prove that any two distinct projective planes in $\mathbb{P}^3$ must intersect in a projective line.
A projective plane in $\mathbb{P}^3$ is of the form $P(U)$ where U is a linear subspace of $\mathbb{R}^4$ of dimension 3. Two distinct 3-dimensional subspaces of $\mathbb{R}^4$ must intersect in a two-dimensional subspace Z because of lemma 0.3 of the lecture notes. The corresponding projective planes therefore intersect in the pline $P(Z)$.

3. Define a Riemannian chart (P, g) by $P = (0,6)^3$ and g is given by $g_{12} = g_{21} = g_{23} = g_{32} = 0$ and $g_{11} = g_{22} = g_{33} = 1$ and $g_{13}(x,y,z) = g_{31}(x,y,z) = \frac{y}{3}$.

(a) Define curves $\alpha, \beta : (-1,1) \to P$ by $\alpha(t) = ((1-t)^2, 1-\sin(t), 1-t)$ and $\beta(t) = (e^{2t}, e^{-t}, e^{-t})$. Prove that α and β cannot both be geodesics with respect to the metric g.
Since $\alpha(0) = \beta(0)$ and $\dot{\alpha}(0) = \dot{\beta}(0)$ the uniqueness theorem of geodesics says that $\alpha = \beta$ on at least a small neighborhood of 0 in case both are geodesics. Looking at the formulas it should be clear that $\alpha(1/n) \neq \beta(1/n)$, so they cannot both be geodesics.

(b) Find the angle between the curves α and β at their intersection point $\alpha(0) = \beta(0) = (1,1,1)$.
The angle is 0 because the angle between the curves is the angle between their tangent vectors $\dot{\alpha}(0) = \dot{\beta}(0)$ with respect to the inner product $g(1,1,1)$ and some chosen orientation. However the angle formula shows that the angle between two equal vectors is always 0 regardless of the orientation.

(c) Find the length of the curve $\gamma : (-1,1) \to P$ given by $\gamma(t) = (1, e^t, e^t)$ with respect to g.
Since $\dot{\gamma}(t) = e^t e_2 + e^t e_3$ we have $g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t)) = e^{2t} g_{22}(\gamma(t)) + e^{2t} g_{33}(\gamma(t)) + 2e^{2t} g_{23}(\gamma(t)) = 2e^{2t}$. Therefore $L(\gamma) = \int_{-1}^{1} \sqrt{g(\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t))}dt = \int_{-1}^{1} \sqrt{2e^{2t}}dt = \sqrt{2}\int_{-1}^{1} e^t dt = \sqrt{2}(e - \frac{1}{e})$.

(d) Is $F : P \to P$ given by $F(x,y,z) = (3-(x-3), 3-(y-3), 3-(z-3))$ a Riemannian isometry from (P,g) to (P,g)?
For any p derivative $dF(p)$ is the linear map $dF(p)(x,y,z) = (-x,-y,-z)$ so $dF(p) = -id$ for all p. We should compare for any two vectors v, w the numbers $g(F(p))(dF(p)v, dF(p)w)$ with $g(p)(v,w)$. If we choose $v = e_1$ $w = e_3$ and $p = (2,2,2)$ then $F(p) = (4,4,4)$ and $g(F(p))(dF(p)v, dF(p)w) = g(4,4,4)(-v,-w) = g(4,4,4)(v,w) \neq g(3,3,3)(v,w)$ because $g(x,y,z)(e_1, e_3) = g_{13}(x,y,z) = \frac{y}{3}$.