Mock exam Geometry 2022

1. Suppose $n \geq 2$ is a positive integer and imagine an n-simplex $J = [v_0, \dots, v_n]$ in $\mathbb{R}^n$. K is the simplicial complex in $\mathbb{R}^n$ consisting of all the faces of J that have dimension 2 or less.

   (a) Write down the Euler characteristic of K as a function of n.

   (b) For any $i \in \{0, 1, \dots, n\}$, denote by $R_i : \mathbb{R}^n \to \mathbb{R}^n$ the affine reflection in the affine hyperplane spanned by the $(n-1)$-dimensional face of J that does NOT contain v_i. Prove that for any $i, j \in \{0, 1, \dots, n\}$ the composition $R_i \circ R_j$ is an affine rotation.

   (c) Define $L = K \cup \{R_0(\sigma) | \sigma \in K\}$. Prove that L is a simplicial complex.

   (d) Find an explicit example of a simplex J as above such that $|L|$ is not a convex polyhedron in $\mathbb{R}^n$.

2. Define $f(x, y) = x^2 - y - 1$. Homogeneous coordinates in $\mathbb{P}^2$ are taken with respect to the standard basis of $\mathbb{R}^3$. Polarity is taken with respect to the standard inner product on $\mathbb{R}^3$.

   (a) Find a non-zero polynomial in three variables $F(x, y, z)$ such that $P(X(f) \times \{1\}) \subset P(X(F)) \subset \mathbb{P}^2$.

   (b) Give the homogeneous coordinates of a point in $P(X(F)) \setminus P(X(f) \times \{1\})$.

   (c) Compute the polar of the projective line through the points $[1:1:1]$ and $[1:2:1]$ in $\mathbb{P}^2$.

   (d) Prove that any two distinct projective planes in $\mathbb{P}^3$ must intersect in a projective line.

3. Define a Riemannian chart (P, g) by $P = (0, 6)^3$ and g is given by $g_{12} = g_{21} = g_{23} = g_{32} = 0$ and $g_{11} = g_{22} = g_{33} = 1$ and $g_{13}(x, y, z) = g_{31}(x, y, z) = \frac{y}{3}$.

   (a) Define curves $\alpha, \beta : (-1, 1) \to P$ by $\alpha(t) = ((1-t)^2, 1 - \sin(t), 1 - t)$ and $\beta(t) = (e^{-2t}, e^{-t}, e^{-t})$. Prove that α and β cannot both be geodesics with respect to the metric g.

   (b) Find the angle between the curves α and β at their intersection point $\alpha(0) = \beta(0) = (1, 1, 1)$.

   (c) Find the length of the curve $\gamma : (-1, 1) \to P$ defined by $\gamma(t) = (1, e^t, e^t)$ with respect to g.

   (d) Is $F : P \to P$ given by $F(x, y, z) = (3-(x-3), 3-(y-3), 3-(z-3))$ a Riemannian isometry from (P, g) to (P, g)?