Contents

Chapter 0

Introduction and preliminaries

My guiding principles for writing these notes and also for doing mathematics are the following. Every concept, construction and definition should come with a motivation, a story, plenty of interesting examples and a computer implementation.

Some main questions we seek to answer are:

1. How to build, describe and manipulate objects arbitrary dimensions? (Euclidean geometry)
2. How to deal with the concept of points at infinity? (Projective geometry)
3. How to find the shortest path between two points in a curved space? (Riemannian geometry)

A unifying theme in each of the geometries that we will investigate is to decide which transformations preserve the geometry. What is the structure of this group of transfomations? Much of the essence of the geometry is hidden in these transformations. In Euclidean and Riemannian geometry the relevant transformations are those that preserve distance. In projective geometry it is rather the projective transformations.

Rough tentative schedule:

Week 1: Euclidean space, simplices, simplicial complexes. Riemannian metrics, length Recall of MVA.

Week 2: Linear algebra. perspective mapping, Riemannian metric, angles.

Week 3: Euclidean isometries. Riemannian pull-back metrics

Week 4: Eucl. Isometries. Variational principle Geodesics

Week 5: Eucl. Isometries, quaternions, Christoffel symbols, geod equation

Week 6: Eucl. Simpl complexes Euler char., homology Projective geometry

Week 7: Eucl. Simpl complexes 2 homology, Projective conics, quadrics

Week 8: Projective duality, Riemannian Recap.

0.1 Linear algebra review

So you want to get good at mathematics? My advice is: invest in linear algebra! It is at the core and very foundation of many if not all subjects in mathematics, not just geometry. In this section we will recall some basic notions of linear algebra and perhaps introduce a few slightly less basic ones that will be useful in our study of geometry.

We will think of vectors as arrows or directions or points in space but conversely we will also use the axioms of a vector space as an algebraic foundation of what we mean by space. The vectors can be added and scaled. In this section we will assume the scalars belong to some arbitrary field $\mathbb{F}$. Usually we will take $\mathbb{F} = \mathbb{R}$ but occasionally we will also allow complex numbers $F = \mathbb{C}$ or at the other extreme only allow the scalars 0 and 1 so $F = F_2 = \mathbb{Z}/2\mathbb{Z}$ (the field with two elements).

The most basic example of a vector space is $\mathbb{F}$ itself.

Perhaps a dual way vectors and vector spaces appear to us in practice is as spaces of functions. For any set S consider $\mathrm{Fun}(S) = \{f : S \to \mathbb{F}\}$, the set of $\mathbb{F}$-valued functions on S. It is a vector space with respect to the addition $(f + g)(s) = f(s) + g(s)$ and scalar multiplication $(\lambda f)(s) = \lambda f(s)$.

The span of a set $S \subset V$ is the set of all linear combinations of elements of S, notation $\underline{S}$. So $\underline{S} = \{\sum_{s\in S} f(s)s | f \in \mathrm{Fun}(S)\}$. For this course we will mostly assume there exists some finite set S such that $V = \underline{S}$.

The number of elements of a set will be denoted $\#S$ and the dimension of V will be $\dim V = \min_{S:\underline{S}=V} \#S$. In other words the least number of vectors that span the whole space.

A related concept is linear independence. A finite set of vectors $S \subset V$ is called linearly independent if for any $f \in \mathrm{Fun}(S)$ we have $\sum_{s\in S} f(s)s = 0 \to f = 0$. In other words, there cannot be any linear relation between the members of S.

By a basis of a vector space we mean an *ordered* sequence of vectors $b_1, \dots b_n$ in V that is linearly independent and spans V. Necessarily $n = \dim(V)$. The crucial feature of a basis is that any vector can be written uniquely as a linear combination of basis vectors. In other words for all $v \in V$ there exist unique coefficients $c^1, \dots c^n \in \mathbb{F}$ such that $v = \sum_{i=1}^n c^i b_i$. Be careful that we sometimes use superscripts to mean powers but usually and in this case the superscript c^2 is just a superscript and does not mean the square of c. The meaning should always be clear from the context.

A linear subspace $W \subset V$ is a subset of V that is closed under scalar multiplication and addition. Using the scalar multiplication and addition of V the subspace W becomes a vector space in its own right.

Another useful construction of vector spaces is the free vector space spanned by some set S. We will call this $\mathbb{F}^S$ in analogy to the notation $\mathbb{R}^n$. By definition the elements of $\mathbb{F}^S$ are finite formal linear combinations $\sum_{i=0}^k \lambda_i s_i$ for any $k \in \mathbb{N}$ and $\lambda_i \in \mathbb{F}$ and $s_i \in S$. Addition and scalar multiplication are done by simply adding the expressions by. For example if $S = \{\text{apples, oranges, bananas}\}$ then we could say

$$(2 \text{ apples} + 4 \text{ oranges}) + 5(\text{bananas} - \text{ oranges}) = 2 \text{ apples} - \text{ oranges} + 5 \text{ bananas} \in \mathbb{F}^S$$

If this seems silly to you then consider polynomials in a variable x and notice that these are much like $\{1, x, x^2, \dots\}^{\mathbb{F}}$. Linear combinations of monomials.

The quotient of a vector space V by a subspace W is the quotient of Abelian groups $V/W = \{v+W|v \in V\}$. Here and in much that follows we will write $v + W = \{v + w|w \in W\}$. The quotient is a vector space, not just an Abelian group and can also be thought of as working with vectors modulo elements in W.

A function between vector spaces $L : V \to W$ is called linear if it respects the linear algebra operations. So for all $v, v' \in V$ and all $\lambda \in \mathbb{F}$ we have $L(v + \lambda v') = L(v) + \lambda L(v')$. The kernel (null-space) of a linear map is defined by $\ker L = \{v \in V|L(v) = 0\}$. The image is $L(V) = \{L(v)|v \in V\}$. The kernel is a linear subspace of V and the image is a linear subspace of W.

Finally an important example of a vector space is the dual vector space $V^* = \{f \in \mathrm{Fun}(V)|\text{f is linear}\}$. Since $\mathrm{Fun}(V)$ was a vector space we can check that V^* is a linear subspace of $\mathrm{Fun}(V)$. The elements of V^* are sometimes referred to as dual vectors. Given a basis $b_1, \dots b_n$ of V we have a dual basis $\beta^1 \dots \beta^n$ of V^* where we define $\beta^i \in V^*$ as follows. The number $\beta^i(v)$ is the coefficient of b_i when expressing v in the basis $b_1 \dots b_n$. So we could also say that $\beta^i(b_j) = \delta^i_j$, where δ is the Kronecker delta function.

A bilinear map is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that for any $v, v', w \in V$ and $\lambda \in \mathbb{F}$ we have $\langle w, v + \lambda v' \rangle = \langle w, v \rangle + \lambda \langle w, v' \rangle$ and $\langle v + \lambda v', w \rangle = \langle v, w \rangle + \lambda \langle v', w \rangle$. A bilinear map is symmetric when $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.

When $\mathbb{F} = \mathbb{R}$ a symmetric bilinear map is said to be an inner product if $\langle v, v \rangle \geq 0$ with equality only when $v = 0$. The corresponding norm is defined by $|.| : V \to [0, \infty)$ with $|v| = \sqrt{\langle v, v \rangle}$ where it is understood we are to take the non-negative value of the square root. The norm actually determines the inner product by the polarization identity $\langle v, w \rangle = \frac{|v+w|^2 - |v|^2 - |w|^2}{4}$. Also, the Cauchy-Schwartz inequality says $|\langle v, w \rangle| \leq |v||w|$ for all $v, w \in V$.

For any bilinear map on V and $w \in V$ we introduce $\langle w, \cdot \rangle \in V^*$ by $\langle w, \cdot \rangle(v) = \langle w, v \rangle$. When the bilinear form is an inner product any element of V^* is of this form.

A basis $b_1, \dots, b_n$ of V is said to be orthonormal with respect to inner product $\langle ., . \rangle$ if $\langle b_i, b_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Starting with any given basis $c_1, \dots c_n$ we can use the Gram-Schmidt algorithm to construct an orthonormal basis $b_1, \dots b_n$ as follows. First define $U(v) = \frac{v}{|v|}$ and set $b_1 = U(c_1)$. Suppose we already found $b_1, \dots b_{k-1}$ then we find b_k by

$$b_k = U\big(c_k - \sum_{j=1}^{k-1} \langle c_k, b_j \rangle b_j\big)$$

Definition 0.1. (Orientation)
We say two bases on a vector space V are equivalent if the determinant of the linear map taking one basis to the other is positive. By an orientation $\mathcal{O}$ on V we mean an equivalence class of bases in the above sense.

Lemma 0.2. (Dimensions of intersections of linear subspaces)
For linear subspaces $U, W \subset V$ we have

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W)$$

Proof. □

Definition 0.3. (Affine subspace, line, (hyper)plane)
An affine subspace $\mathcal{P} \subset V$ is a subset of the form $\mathcal{P} = q + P = \{q + p | p \in P\}$, for some linear subspace $P \subset V$ and $q \in V$. The linear subspace P is called the **direction** *of $\mathcal{P}$. The dimension of $\mathcal{P}$ is the dimension of its direction. The co-dimension is $n - \dim(\mathcal{P})$.*

When $\mathcal{P}$ has codimension 1 it is referred to as an affine hyperplane. The 1 and 2 dimensional cases are called affine lines and planes. An interesting example of an affine hyperplane is $\mathcal{U} \subset \mathbb{R}^n$ containing all the standard basis vectors $e_1, \dots e_n$. Its direction is $U = \overline{\{e_i - e_j | i, j \leq n\}}$ and $\mathcal{U} = e_1 + U$.

An alternative way to think of affine hyperplanes is to pick some $\lambda \in \mathbb{F}$ and a $f \in V^*$ and consider $\mathcal{P} = \{v \in \mathbb{R}^n | f(v) = \lambda\}$. This is indeed an affine subspace and its direction is $\ker f$.

In our example $\mathcal{U}$ we could take $f = \varepsilon^1 + \varepsilon^2 \cdots + \varepsilon^n \in (\mathbb{R}^n)^*$, and $\lambda = 1$ where ε^i denotes the i-th dual basis vector. In case we have an inner product we may also write $\varepsilon^i(v) = \langle e_i, v \rangle$.

As an illustration of how results on linear subspaces extend to affine subspaces we consider the dimension of intersections.

Lemma 0.4. (Dimensions of intersections of affine subspaces)
For affine subspaces $\mathcal{U}, \mathcal{W} \subset \mathbb{R}^n$ with directions U, W we have

$$\dim(\mathcal{U} \cap \mathcal{W}) = \dim(\mathcal{U}) + \dim(\mathcal{W}) - \dim(U + W)$$

unless $\mathcal{U} \cap \mathcal{W} = \emptyset$.

Exercises

1. (a)
   (b) Show that every vector space has precisely two orientations.

0.2 Multivariable analysis review

Curve, derivative, C^1-function, vector field, ODE theorem.

Exercises

1. (a)

Chapter 1

Euclidean geometry

Following Descartes we use numbers as a foundation for geometry. In more modern terms, we replace Euclid's axioms by the axioms of a vector space. By n-dimensional Euclidean space we mean $\mathbb{R}^n$ with the standard inner product $\langle \cdot, \cdot \rangle$ and orientation containing the standard basis $e_1, e_2, \dots, e_n$. While it is convenient to speak about Euclidean space in terms of $\mathbb{R}^n$, we should remember that for the purposes of geometry the origin is not considered a special point. Also the basis $e_1, \dots e_n$ is no more special than any other orthonormal basis in the same orientation.

1.1 Polytopes, angles and spheres

Convex polytopes and in particular simplices are is a direct generalization of the idea of a line segment $[a, b] = \{ta + (1-t)b | t \in [0, 1]\}$. Notice that in constructing a line segment connecting points a and b we take all possible weighted averages of the points a and b. Here the weight of a is t and the weight of b is $1 - t$. The midpoint is the special case with equal weights and corresponds to the actual average. More symmetrically we could describe a point on the interval as $w_a a + w_b b$ where $w_a + w_b = 1$ and $w_a, w_b \in [0, 1]$ is on the same segment.

In the same way a convex polytope contains all weighted averages of some finite set of points.

Definition 1.1. (Convex polytope)
For any finite $S \subset \mathbb{R}^n$ define the convex polytope

$$[S] = \{\sum_{s \in S} w_s s | \forall s \in S\, , w_s \in [0, 1]\, , \sum_{s \in S} w_s = 1\}$$

We often use the abbreviation $[p_0, \dots, p_k] = [\{p_0, \dots, p_k\}]$.

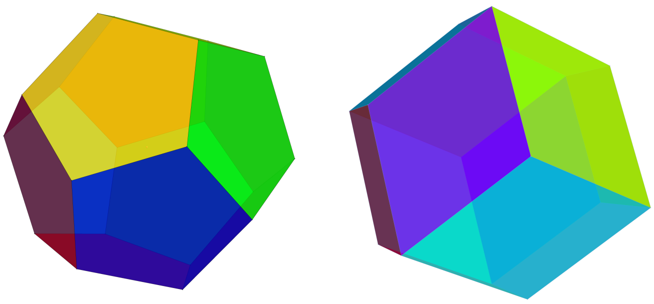

Figure 1.1: Left: Euclid's dodecahedron. Right: Kepler's rhombic dodecahedron.

Besides the line segments, examples of convex polytopes include the n-dimensional cubes

$$[a, b]^n = [\{c_1, c_2, \dots, c_n | c_i \in \{a, b\}\}]$$

and the orthoplexes $[\{\pm e_i | i = 1 \dots n\}]$. Building on the three-dimensional cube $[\{\pm 1, \pm 1, \pm 1\}]$ both the dodecahedra of Euclid and Kepler can be described as

$$D_E = [\{\pm 1, \pm 1, \pm 1\} \cup \{0, \pm\tau, \pm\tau^{-1}\}, \cup\{\pm\tau^{-1}, 0, \pm\tau\}, \cup\{\pm\tau, \pm\tau^{-1}, 0\}]$$

$$D_K = [\{\pm 1, \pm 1, \pm 1\} \cup \{0, 0, \pm 2\}, \cup\{0, \pm 2, 0\}, \cup\{\pm 2, 0, 0\}]$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. The signs of the coordinates above can all be chosen independently so that the set defining D_E has 20 elements and D_K is defined by 14 vertices.

The most important class of convex polytopes are the simplices and we will use them as building blocks for more complicated shapes.

Definition 1.2. (Simplex)
We say $[v_0, v_1, \dots v_k]$ is a k-dimensional **simplex** *if the set $\{v_1 - v_0, v_2 - v_0, \dots, v_k - v_0\}$ is linearly independent. By convention we allow the empty set to be a simplex of dimension -1. If $[S]$ is a simplex and $F \subset S$, we say $[F]$ is a* **face** *of $[S]$.*

Defining angles between vectors in higher dimensions takes a little care as one needs to keep track of the orientation of the plane spanned by the vectors. For example, what is the angle between e_3 in and e_5 in $\mathbb{R}^5$? The inner product tells us these vectors are perpendicular but is the angle $\frac{\pi}{2}$ or $-\frac{\pi}{2}$? The distinction is important if we want to be able to add angles consistently modulo 2π.

For a single pair of vectors there is no way to decide the sign of the angle. We need a reference frame, or rather an orientation on our plane P to define the angle. In practice it is convenient to use Gram-Schmidt on any positively oriented basis c_1, c_2 of P to make an orthonormal basis $b_1 = U(c_1), b_2 = U(c_2 - \langle b_1, c_2 \rangle b_1)$. We can then decide angles in terms of complex numbers using the orientation and inner product preserving map $F : P \to \mathbb{C}$ defined by $F(b_1) = 1$ and $F(b_2) = i$.

Definition 1.3. (Angle)
Consider an oriented 2-dimensional linear subspace $P \subset \mathbb{R}^n$. The angle $\angle(u, v)$ between unit vectors $u, v \in P$ is defined in terms of an orientation and inner product preserving linear map $F : P \to \mathbb{C}$ by

$$e^{i\angle(u,v)} = F(v)\overline{F(u)}$$

For distinct points A, B, O in an affine an 2-plane $\mathcal{P}$ with oriented direction P define

$$\angle AOB = \angle\left(\frac{A - O}{|A - O|}, \frac{B - O}{|B - O|}\right)$$

Lemma 1.4. (Addition of angles)

1. *The angle $\angle(u, v)$ does not depend on the choice of F in Definition 1.3 and is well defined modulo 2π.*
2. *For unit vectors u, v, w vectors in an oriented 2-plane $P \subset \mathbb{R}^n$ we have: $\angle(u, v) + \angle(v, w) = \angle(u, w) \mod 2\pi$.*

Proof. Suppose $G : P \to \mathbb{C}$ is another orientation and inner product preserving linear map. Then we claim $\varphi = G \circ F^{-1}$ is multiplication by some complex number z of unit norm (Exercise!). It follows that $G(v)\overline{G(u)} = F(v)\overline{F(u)}$. We know from complex analysis that $\angle(u, v) = \log e^{i\angle(u,v)}$ is well-defined up to integer multiples of 2π.

The final part follows from taking the complex logarithm on both sides of

$$e^{i(\angle(u,v)+\angle(u,w))} = F(w)\overline{F(v)}F(v)\overline{F(w)} = F(w)\overline{F(u)} = e^{i\angle(u,w)}$$

□

Definition 1.5. (Ball, sphere)
The n-sphere and n-ball are defined by $S^n_{c,r} = \{x \in \mathbb{R}^n : |x - c| = r\}$ and closed ball $\bar{B}^n_{c,r} = \{x \in \mathbb{R}^n : |x - c| \le r\}$

Definition 1.6. (Inversion in a sphere)
Inversion is the map $\mathcal{I}_{c,r} : \mathbb{R}^n - \{c\} \to \mathbb{R}^n$ defined by $\mathcal{I}_{c,r} = r\frac{x-c}{|x-c|} + c$.

Exercises

1. (a)

1.2 Simplicial complexes

Definition 1.7. (Simplicial complex)
A finite set K of simplices in $\mathbb{R}^n$ is called a **simplicial complex** *if every face of a simplex of K is also in K and the intersection between any two simplices in K is a face of both. Define K_ℓ to be the set of all ℓ-simplices in K. The maximum dimension of the simplices of K is called the dimension of K. Define the underlying set to be $|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n$.*

Definition 1.8. (Euler characteristic)
The Euler characteristic $\chi(K)$ of a simplicial complex is defined to be $\chi(K) = \sum_{d=0}(-1)^d \# K_d$.

Definition 1.9. (Collapse, equivalence)
We say a nonempty simplex $\tau \in K$ is a free face if there is a unique $\sigma \in K$ that strictly contains it. In this case $L = K - \{\sigma, \tau\}$ is called a **collapse** *of K.*

Finally K is (simple homotopy) **equivalent** *to M if there is a finite sequence of simplicial complexes $K^{(j)}$ such that $K = K^{(0)}$ and $M = K^{(m)}$ and either $K^{(j)}$ is a collapse of $K^{(j+1)}$ or vice versa.*

Lemma 1.10. *If two simplicial complexes are equivalent they must have the same Euler characteristic.*

Proof. If $L = K - \{\sigma, \tau\}$ is a collapse of K then we must have $\dim \sigma = \dim \tau + 1$ because otherwise there will be another face of σ that also contains τ. This means that $\chi(K) = \chi(L)$ as we remove precisely one odd-dimensional and one even-dimensional simplex. A general simple homotopy equivalence is a finite sequence of collapses and their inverses so the Euler characteristic will not change. □

Definition 1.11. (Simplicial surface)
A simplicial surface is a two-dimensional simplicial complex such that every 1-simplex is the face of precisely two 2-simplices and every 0-simplex is the face of at least three 1-simplices.

The (discrete Gauss) curvature $\kappa(p)$ at 0-simplex p is 2π minus the sum of the angles of the triangles that meet there.

Theorem 1.12. (simplicial Gauss-Bonnet)
For any simplicial surface we have:

$$\sum_{p \in \Sigma_0} \kappa(p) = 2\pi\chi(\Sigma)$$

Proof. Grouping the angles by the triangle they belong to and taking for granted that the three angles in any triangle sum to π we obtain

$$\sum_{p \in \Sigma_0} \kappa(p) = 2\pi \# \Sigma_0 - \pi \# \Sigma_2 = 2\pi\chi(\Sigma)$$

The last equality is a consequence of $3\#\Sigma_2 = 2\#\Sigma_1$. □

Definition 1.13. (ℓ-chains)
For a simplicial complex K and an integer $\ell \geq 0$ define $C_\ell(K)$ to be the free vector space over $\mathbb{F}_2$ spanned by all ℓ-simplices in K.

Definition 1.14. (Boundary maps)
For every ℓ define a linear map $\partial_\ell : C_\ell(K) \to C_{\ell-1}(K)$ by

$$\partial_\ell([S]) = \sum_{s \in S} [S - \{s\}]$$

Definition 1.15. (Homology)
Define ℓ-th homology of simplicial complex K to be

$$H_\ell(K) = \ker \partial_\ell / \operatorname{im} \partial_{\ell+1}$$

Lemma 1.16. *If two simplicial complexes are equivalent they must have the same homology.*

Proof. As with the Euler characteristic it suffices to consider the case where $L = K - \{\sigma, \tau\}$ is a collapse of K. Suppose $\dim \sigma = s + 1$ so that the free face τ is s dimensional. $H_\ell(K)$ is defined in terms of $C_i(K)$ for $\ell - 1 \leq i \leq \ell + 1$ so unless $s - 1 \leq \ell \leq s + 2$ we will have $H_\ell(K) = H_\ell(L)$. We treat the remaining four cases separately.

If $s = \ell + 1$ then $\mathrm{im}\partial_{\ell+1}$ in K might not equal $\mathrm{im}\partial_{\ell+1}$ in L. But in fact it does because even though τ is missing we still have $\partial\tau = \partial(\tau - \partial\sigma)$ and $\tau - \partial\sigma$ consists of all s-dimensional faces of σ except τ.

If $s = \ell$ then we note that $\partial\tau \neq 0$

□

Exercises

1. (a)

1.3 Euclidean isometries

Definition 1.17. *A* **Euclidean isometry** *is a bijection* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *that preserves Euclidean distance:* $\forall x, y \in \mathbb{R}^n : |x - y| = |\varphi(x) - \varphi(y)|$. *The set of all isometries is called the Euclidean group, notation* $E(n)$. *Also, the set of all isometries sending* 0 *to itself is called the orthogonal group, notation* $O(n)$.

Definition 1.18. (Translation, reflection, rotation)

1. *The translation* $T_v \in E(n)$ *by vector* $v \in \mathbb{R}^n$ *is defined by* $T_v(p) = p + v$.
2. *The (affine) reflection* $R_\mathcal{M} \in E(n)$ *in affine hyperplane* $\mathcal{M}$ *(called the mirror) is defined by* $\frac{1}{2}(R_\mathcal{M}(p)+p) = m$ *where* m *is the point on* $\mathcal{M}$ *closest to* p.
3. *An affine rotation is the composition of two reflections whose mirrors have non-empty intersection.*

If $\mathcal{M} = m + u^\perp$ with $|u| = 1$ we have $R_\mathcal{M}(p) = p - 2\langle p - m, u\rangle u$.

If we identify the Euclidean plane $\mathbb{R}^2$ with $\mathbb{C}$ by sending e_1 to 1 and e_2 to i then a rotation is a map $z \mapsto e^{i\theta} z$ and a reflection is $z \mapsto \bar{z}$ and a translation is $z \mapsto z + b$. A general element of $E(2)$ is of the form $z \mapsto e^{i\theta} z + b$ or $z \mapsto e^{i\theta}\bar{z} + b$ for some $\theta \in (-\pi, pi)$ and $b \in \mathbb{C}$.

Lemma 1.19. *The elements of* $O(n)$ *are linear maps that satisfy* $\langle\varphi(v), \varphi(w)\rangle = \langle v, w\rangle$ *for all* $v, w \in \mathbb{R}^n$.

Proof. Suppose $\psi \in O(n)$ so ψ preserves distances and $\psi(0) = 0$. By the polarization identity ψ preserves the inner product (Exercise!) so we just need to show it is a linear map. Since the standard basis is orthonormal, so is the basis $f_1, \dots f_n$ defined by $f_i = \psi(e_i)$. Recall that any vector can be written uniquely as $v = \sum_i \langle v, e_i\rangle e_i$ and $\psi(v) = \sum_i \langle\psi(v), f_i\rangle f_i = \sum_i \langle v, e_i\rangle f_i$ So for any $a \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$ we find linearity:

$$\psi(av + w) = \langle av + w, e_i\rangle f_i = a\langle v, e_i\rangle f_i + \langle w, e_i\rangle f_i = a\psi(v) + \psi(w)$$

□

Theorem 1.20. (The structure of $E(n)$)

1. *Any Euclidean isometry* F *can be written uniquely as* $F = T_v \circ \varphi$ *for some* $v \in \mathbb{R}^n$ *and* $\varphi \in O(n)$.
2. *For* $\varphi, \psi \in O(n)$ *and* $v, w \in \mathbb{R}^n$ *we have*

$$T_v \circ \varphi \circ T_w \circ \psi = T_{v+\varphi(w)} \circ \varphi \circ \psi$$

Proof. To see existence we write $F = T_{F(0)} \circ T_{-F(0)} \circ F$. Since $\varphi = T_{-F(0)} \circ F$ sends 0 to 0 it is in $O(n)$.

To see uniqueness suppose $F = T_v \circ \varphi = T_{v'} \circ \varphi'$. Composing with $T_{-v'}$ from the left and with φ^{-1} from the right we get $T_{v-v'} = \varphi' \circ \varphi^{-1} \in O(n)$. This forces $v - v'$ because the identity is the only translation that fixes the origin as any element of $O(n)$ should. In turn this also implies $\varphi = \varphi'$.

The second part follows from $(\varphi \circ T_w \circ \varphi^{-1})(x) = \varphi(w + \varphi^{-1}(x)) = \varphi(w) + x = T_{\varphi(w)}(x)$, using the fact that φ is a linear map. To finish the argument we write:

$$T_v \circ \varphi \circ T_w \circ \psi = T_v \circ \varphi \circ T_w \circ \varphi^{-1} \circ \varphi \circ \psi = T_{v+\varphi(w)} \circ \varphi \circ \psi$$

□

Definition 1.21. *Define* $SO(n) = \{\varphi \in O(n) | \det\varphi = 1\}$. *An isometry* $F \in E(n)$ *is called orientation preserving if* $F = T \circ \varphi$ *with* $\varphi \in SO(n)$.

Theorem 1.22. (Linear reflections generate $O(n)$)
If for $L \in O(n)$ *there exists a linear subspace* $U \subset V$ *of codimension* c *such that* $\forall u \in U : L(u) = u$, *then* L *is the composition of at most* c *linear reflections in* $O(n)$.

Proof. We argue by induction on the codimension c. When $c = 0$ we must have $L = id_{\mathbb{R}^n}$. For the induction step, suppose L fixes a codimension c subspace U and $Lv = w \neq v$. Reflection in the mirror M through 0 equidistant to v and w is a linear map R_M. Moreover $R_M \circ L$ fixes the codimension $c - 1$ subspace spanned by v and U. This is because $v - w$ is orthogonal to U (why?) so by the induction hypothesis the proof is complete. □

Lemma 1.23. *Any non-trivial element of* $L \in SO(3)$ *is the composition of precisely two reflections. The intersection of the two mirrors is called the axis* A.

$L \in SO(3)$ has determinant 1 and is the composition of two reflections. When the reflection planes do not coincide there is 1-dimensional eigenspace of eigenvalue 1. The other two eigenvalues must be inverses and also complex conjugates because the characteristic polynomial has real coefficients. Therefore the other two eigenvalues must be $e^{\pm i\theta}$.

Definition 1.24. (Quaternions)
The quaternions $\mathbb{H}$ are defined as a 4-dimensional vector space with basis $1, i, j, k$ and bilinear, associative product multiplication determined by Hamilton's relations

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 \quad \mathrm{ij} = \mathrm{k} \quad \mathrm{jk} = \mathrm{i} \quad \mathrm{ki} = \mathrm{j}$$

If $q = w + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} \in \mathbb{H}$ the conjugate is $\bar{q} = w - x\mathrm{i} - y\mathrm{j} - z\mathrm{k}$. The imaginary part of q is $x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$ and the real part is w. The norm is defined as $|q| = q\bar{q} = w^2 + x^2 + y^2 + z^2 \geq 0$.

Calculus books still use $\mathrm{i}, \mathrm{j}, \mathrm{k}$ to denote the unit vectors in $\mathbb{R}^3$ and this originated from Hamilton's work. If we identify vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ with the imaginary quaternion $q_v = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} \in \mathbb{H}$ then we have

$$q_v q_u = -v \cdot u + q_{v \times u}$$

This means that vectors $v \perp u$ if and only if $q_v q_u + q_u q_v = 0$ because $\bar{q}_v = -q_v$.

Now assume u is a unit vector so that $q_u \bar{q}_u = 1$. Then $q_u^2 = -1$ and $v \perp u$ if and only if $q_v = q_u q_v q_u$. Notice that the left hand side is again a purely imaginary quaternion. This means there is a map $s : \mathbb{R}^3 \to \mathbb{R}^3$ given by $v \mapsto q_u q_v q_u = q_{s(v)}$. We claim $s = R_{u^\perp}$ the reflection in the plane orthogonal to unit vector u. Indeed s fixes vectors in the plane $u^\perp$ by the above discussion and $q_{s(u)} = q_u^3 = -q_u = q_{-u}$ since $q_u^2 = -1$.

For two unit vectors u, t composing the reflections $R_{u^\perp}$ and $R_{t^\perp}$ we get a map sending q_v to $q_t q_u q_v q_u q_t$. Now $q_t q_u = (q_u q_t)^{-1}$ since both u, t are unit vectors. The product $q_t q_u$ is also a unit quaternion so all rotations are written in quaterions as conjugation by a unit quaternion.

Theorem 1.25. *The map $\rho : \mathbb{H} \to SO(3)$ defined by $\rho(q) = qp\bar{q}$ is a surjective group homomorphism with kernel generated by ± 1.*

As an application of all this, try to realize the symmetries of the 3d Platonic solids as subsets of the unit sphere in H. Argue that these subsets must be 4d Platonic solids in their own right.

Quaternions to describe 3D rotations and reflections. Symmetry group of dodecahedron as a set of unit quaternions. Voronoi cells in the space of quaternions. Intersect the Voronoi cells with the unit quaternions to get the vertices of the 120-cell. Take the convex hull to get the 120 cell. Consider the cell around the unit element 1. Why is it a dodecahedron?

Exercises

1. umm
2. (a) Any Euclidean isometry is the composition of at most $n + 1$ affine reflections.
3. um

Chapter 2

Projective geometry

The main question we want to tackle in this chapter is how to make geometrical sense of points at infinity.

2.1 Perspective drawing

To set the stage for projective geometry consider standing in $\mathbb{R}^3$ with your head at the origin and your feet on the ground plane $G = \{(x, y, -1)|x, y \in \mathbb{R}\}$. Suppose you want to draw the things you see precisely as they appear on screen $S = \{(x, 1, z)|y, z \in \mathbb{R}\}$ then we define a projection π onto S by $\pi(x, y, z) = (\frac{x}{y}, 1, \frac{z}{y})$, which is defined as long as $y \neq 0$. The idea behind π is to send the point $p = (x, y, z) \in \mathbb{R}^3$ to the unique point on the intersection of S with the line connecting p to our eye at the origin.

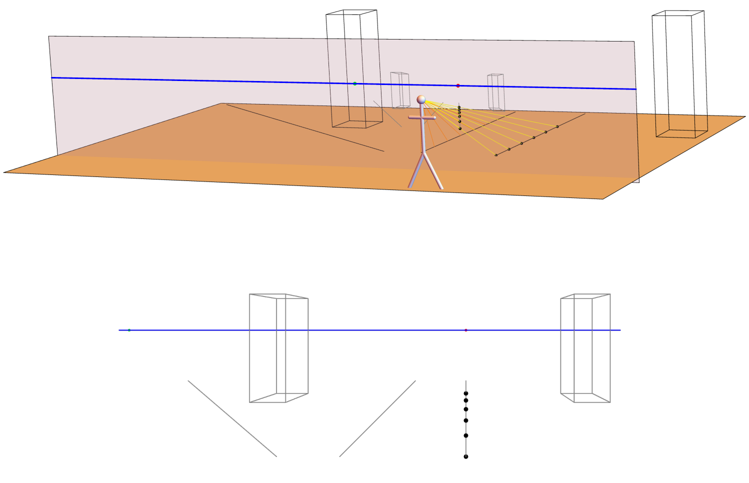

Figure 2.1: Perspective drawing in Gray.

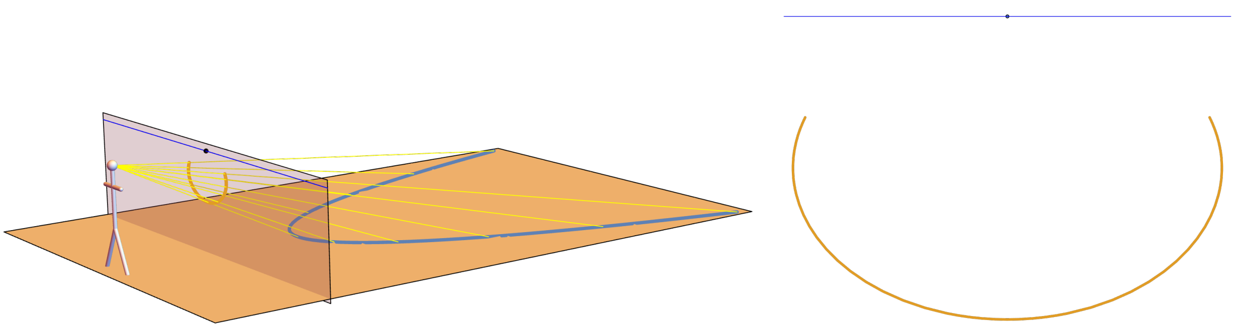

Figure 2.2: Perspective drawing in Gray.

Exercises

1. (a)

2.2 Projective geometry

Instead of intersecting the rays of light with a screen we study the rays themselves. This motivates the following definition.

Definition 2.1. (Projective space)
For any vector space V define $P(V) = \{\underline{v} | 0 \neq v \in V\}$ to be the set of 1-dimensional linear subspaces of V.

Occasionally it will be useful to introduce coordinates in $P(V)$ using a basis of V. These are known as homogeneous coordinates.

Definition 2.2. (Homogeneous coordinates)
We say $\underline{v} \in P(V)$ has homogeneous coordinates $[h^1 : h^2 : \cdots : h^n]$ with respect to the basis $b_1 \ldots b_n$ of V if $v = \sum_{i=1}^{n} h^i b_i$.

Any point $p \in P(V)$ has homogeneous coordinates with respect to a basis of V but they are only unique up to multiplication by a scalar. For example using the standard basis of $\mathbb{R}^3$ the homogeneous coordinates $[1 : 2 : 3]$ and $[-3 : -6 : -9]$ describe the same point $p = \underline{e_1 + 2e_2 + 3e_3} \in P(\mathbb{R}^3)$ in the projective plane. As another example, again using the standard basis, in the projective line $P(\mathbb{R}^2)$ there are two types of points: those with coordinates $[x : 1]$ and the special *infinity* point with coordinate $[1 : 0]$. Of course the choice of which point to call special depends on the chosen basis.

Definition 2.3. (Projective subspace, dimension)
The dimension of $P(V)$ to be $\dim(V) - 1$. If $U \subset V$ is a linear subspace we say that $P(U) \subset P(V)$ is a projective subspace of $P(V)$. The $0, 1,$ and 2-dimensional cases are referred to as projetive points, lines and planes (aka, ppoints, plines, pplanes).

It helps to view projective subspaces on an affine slice of projective space. For example a projective line is $P(U)$ with $U \subset V$ a two-dimensional linear subspace. We call it a projective line because the intersection of U with an affine hyperplane is generally an affine line. $P(U)$ is supposed to be the set of rays that connect the points of said affine line to the origin. U is the two-dimensional linear subspace spanned by all those rays through the origin.

Lemma 2.4. (Affine slice)
If $\mathcal{H} = w + H$ is a hyperplane that does not contain the origin of V then there is a bijection $f_{\mathcal{H}} : \mathcal{H} \to P(V) - P(H)$ given by $f_{\mathcal{H}}(x) = \underline{x}$. The inverse map is defined by sending $\underline{v}$ to the single point in the intersection $\underline{v} \cap \mathcal{H}$. Writing $v = \lambda w + h$ this point is $f_{\mathcal{H}}^{-1} = \lambda^{-1} v$.

Proof. If $f_{\mathcal{H}}(x) \subset H$ for some $x \in \mathcal{H}$ then both x and $x - w$ are in H. This implies $-w, w \in H$ and hence $\mathcal{H}$ contains the origin. Furthermore for $x \in \mathcal{H}$ we have $f_{\mathcal{H}}^{-1}(f_{\mathcal{H}}(x)) = x$ because because the intersection $\underline{x} \cap \mathcal{H}$ contains at most one point and x is certainly in it. Conversely $f_{\mathcal{H}}(f_{\mathcal{H}}^{-1}(\bar{v})) = \underline{v}$ because if $\{x\} = \underline{v} \cap \mathcal{H}$ then $\underline{x} = \underline{v}$. The intersection cannot be empty because $\underline{w \cup H} = V$ so $v = \lambda w + h$ for some $h \in H$. Also, $\lambda \neq 0$ because otherwise $v = h \in H$ meaning $\bar{v} \in P(H)$. The point $\lambda^{-1} v = w + \lambda^{-1} h$ is clearly in $w + H = \mathcal{H}$. □

Lemma 2.5. (Two ppoints determine a pline)

1. *For two distinct ppoints $A, B \in P(V)$ there is a unique pline containing both. It is given by $P(A + B)$, where $A + B = \{a + b | a \in A, b \in B\}$.*
2. *More generally, for projective subspaces $P(U), P(W)$ the subspace $P(U + W)$ is the unique projective subspace of $P(V)$ of least dimension containing both.*

Proof. □

Lemma 2.6. (Two plines determine a ppoint)

1. *Two distinct plines $P(U), P(W) \in P(V)$ intersect in a unique ppoint.*
2. *More generally, for projective subspaces $P(U), P(W)$ the intersection $P(U) \cap P(W) = P(U \cap W)$ is a projective subspace with dimension*

$$\dim(P(U) \cap P(W)) = \dim(P(U)) + \dim(P(W)) - \dim(P(U + W))$$

Proof. □

For studying geometry it is important to consider not just the objects but also the geometric transformations between them. In this case these are known as projective transformations. As in affine geometry the projective transformations will preserve intersections of lines but not shapes and sizes.

Definition 2.7. (Projective transformation)
To any injective linear map $L : V \to W$ we associate a map $P(L) : P(V) \to P(W)$ defined by $P(L)(\underline{v}) = \underline{L(v)}$, called the associated projective transformation.

The associated projective transformation just expresses how the linear map sends lines through the origin to other lines through the origin. For example if $V = W = \mathbb{R}^2$ and L permutes the standard basis elements $L(e_1) = e_2$ and $L(e_2) = e_1$ then $P(L)[a : b] = [b : a]$ if we use homogeneous coordinates with respect to the standard basis.

The assumption of injectivity in Definition 2.7 is to make sure L does not map some line to 0. Two linear transformations may yield the same projective transformation. For example $P(17\mathrm{id}_V) = \mathrm{id}_{P(V)} = P(\mathrm{id}_V)$.

Projective geometry deals with those properties of $P(V)$ that are unchanged under projective transformations. Just like choosing a basis or choosing an orientation we can can use a projective transformation to bring our configurations into a convenient form. For example one may wonder how many points in the projective line can be moved to some desired location by a projective transformation. Since a projective line comes from taking $\dim V = n = 2$ we see that we may send any two lines $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$ to any other two lines, just by finding a linear map L such that $L(v)$ and $L(w)$ are as desired. In fact, we get to choose one more point on the line at will. This motivates the definition of general position:

Definition 2.8. (General position)
If $n = \dim V$ then $n+1$ points of $P(V)$ are in general position if ignoring any one of the points we are left with $\mathrm{Span}(b_1), \mathrm{Span}(b_2), \dots \mathrm{Span}(b_n)$ for some basis $b_1, \dots b_n$ of V.

For example when $V = \mathbb{R}^2$ the triples of points $\mathrm{Span}(e_1), \mathrm{Span}(e_2), \mathrm{Span}(e_1 + e_2) \in P(\mathbb{R}^2)$ and $\{\mathrm{Span}(e_1 - e_2), \mathrm{Span}(e_1 + 2e_2), \mathrm{Span}(e_1 - 2e_2)\}$ are in general position.

Lemma 2.9. *If $p_1, \dots p_{n+1}$ are in general position in $P(V)$ and $q_1, \dots q_{n+1}$ are in general position in $P(W)$ then there exists a unique projective transformation $P(L)$ such that $P(L)p_i = q_i$ for all $i \leq n+1$.*

Proof. By assumption there are bases $\mathbf{b}, \mathbf{c}$ of V and W such that $p_i = \mathrm{Span} b_i$ and $q_i = \mathrm{Span} c_i$ for all $i \leq n$. Furthermore we may assume that $p_{n+1} = \mathrm{Span}(\sum_{i=1}^n b_i)$ and $q_{n+1} = \mathrm{Span}(\sum_{i=1}^n c_i)$ (Exercise!). It follows that the unique linear transformation $L \in \mathrm{Hom}(V, W)$ defined by $L(b_i) = c_i$ for $i \leq n$ will also map the sum of the b_i to the sum of the q_i so that the associated projective transformation $P(L)$ will be as claimed. □

To illustrate the proof set $f_1 = e_1 - e_2$ and $f_2 = e_1 + 2e_2$. Then $e_1 - 2e_2 = \frac{4}{3}f_1 - \frac{1}{3}f_2$ so take $g_1 = \frac{4}{3}f_1$ and $g_2 = -\frac{1}{3}f_2$. Then define $L \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ by $Le_i = g_i$. This defines L uniquely and assures us $L(e_1 + e_2) = e_1 - 2e_2$ so that $P(L)$ sends $\mathrm{Span}(e_1), \mathrm{Span}(e_2), \mathrm{Span}(e_1 + e_2)$ to $\mathrm{Span}(e_1 - e_2), \mathrm{Span}(e_1 + 2e_2), \mathrm{Span}(e_1 - 2e_2)$.

Three points on a projective line can be mapped to any other three points but what about four points? We say two quadruple of distinct points (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) have the same cross-ratio if there exists a projective transformation T such that $T(p_i) = q_i$ for all i. Using homgeneous coordinates we can describe the cross ratio of a quadruple of points by the number c such that if we find T such that $T(p_1) = [0 : 1]$, $T(p_2) = [1 : 1]$ and $T(p_3) = [1 : 0]$ then $T(p_4) = [c : 1]$.

Theorem 2.10. (Desargues)
Imagine six distinct points $A_i, B_i \in P(V)$ where $i \in \{0, 1, 2\}$ such that the three projective lines A_iB_i are distinct and meet in $X \in P(V)$. Then the three points of intersection $A_iA_j \cap B_iB_j$ where $i \neq j \in \{0, 1, 2\}$ are colinear.

Proof. We can find vectors $x, a_i, b_i \in V$ such that $X = \mathrm{Span}(x), A_i = \mathrm{Span}(a_i), B_i = \mathrm{Span}(b_i)$ such that $x = a_i + b_i$ for all $i \in \{0, 1, 2\}$ (why?). For any fixed $i \neq j$ the projective lines A_iA_j and B_iB_j cannot coincide because then the lines A_iB_i and A_jB_j would also coincide. Moreover since $a_j + b_j = x = a_i + b_i$ we have $\mathrm{Span}(a_i - a_j) = \mathrm{Span}(b_i - b_j) = A_iA_j \cap B_iB_j$. Finally, $a_0 - a_1 + a_1 - a_2 + a_2 - a_0 = 0$ so the three points $\mathrm{Span}(a_i - a_j)$ must be on a common projective line (exercise!). □

Exercises

1. (a)

2.3 Conics, quadrics and quadratic equations

Throughout this section we will assume $\dim V = 3$ for simplicity.

Definition 2.11. (Symmetric bilinear forms)
A symmetric bilinear form on V is a bilinear map $B : V \times V \to F$ such that $B(v, w) = B(w, v)$, so $B(\lambda u+v, w) = \lambda B(u, w) + B(v, w)$. B is said to be non-degenerate if $\forall v \in V : B(v, w) = 0$ implies $w = 0$.

In fact the function $B(v, v) : V \to \mathbb{R}$ determines the bilinear form B completely since $2B(v, w) = B(v + w, v + w) - B(v, v) - B(w, w)$. A function of the form $B(v, v)$ is often called a quadratic form.

For a basis $\mathbf{b}$ of V we can write $v = \sum_{i=0}^{n} x_i b_i$ and so the quadratic form $B(v, v)$ is expressed as $B(v, v) = \sum_{i=0}^{n} B_{ij} x_i x_j$ where we set $B_{ij} = B(b_i, b_j)$.

Theorem 2.12. (Normal form)
Suppose B is a symmetric bilinear form on V. There exists a basis $\mathbf{b}$ such that if $v = \sum_{i=0}^{n} x_i b_i$ then $B(v, v) = \sum_{i=0}^{p} x_i^2 - \sum_{i=p}^{p+q} x_i^2$ for some $p, q \leq n$. If B is non-degerenate then $n = p + q$.

Proof. Rotate and complete the square. □

Definition 2.13. (Quadrics)
Suppose B is a symmetric bilinear form on V. The set $Q(B) = \{\underline{v} \in P(V) | B(v, v) = 0\}$ is called a quadric in $P(V)$. The quadric is said to be non-singular if B is non-degenerate.

In the special case where $\dim V = 3$ we refer to quadrics as conics for the following reason. For any quadratic equation in two variables $f(x, y) = 0$ we can turn it into a homogeneous equation $F(x, y, z)$ such that $f(x, y) = F(x, y, -1)$. This way we can view the zero set of $f(x, y) = 0$ as in the beginning of this chapter. In suitable coordinates the zero set $F(a, b, c) = 0$ has the form $a^2 + b^2 = c^2$. I.e. it is a double cone! It follows that $f(x, y)$ is the intersection of a cone with a plane.

2.4 Projective duality

The dual space V^* is a vector space in its own right and as such we may do projective geometry in $P(V^*)$. More interestingly the points of $P(V^*)$ correspond to projective hyperplanes in $P(V)$ and vice versa. For example in the projective plane, plines in $P(V)$ correspond to ppoints in $P(V^*)$ and so every statement about the projective plane gives rise to a dual statement by exchanging the roles of points and lines. This is called duality.

Before we begin let us emphasize that even though V^* and V have the same dimension and are isomorphic as vector spaces it is a good habit not to mix the two up lightly as there is no natural (basis independent) isomorphism identifying the two. Identifying V and V^* tacitly chooses an inner product or at least a non-degenerate bilinear form $V \times V^* \to \mathbb{R}$.

In contrast there is a good way to identify the dual of the dual $V^{**} = (V^*)^*$ with the original V. Indeed define $\Psi : V \to V^{**}$ by $\Psi(v)(f) = f(v)$ for all $v \in V, f \in V^*$. Then Ψ is clearly linear and also $\ker \Psi$ must be 0 as a map $f \in V^*$ is 0 if its value on V is 0. Inversely we know a vector $v \in V$ is determined if we know values of all the $f \in V^*$ on v. So we set $z = \Psi^{-1}(\phi) \in V$ to be the unique vector defined by $f(z) = \phi(f)$. In what follows we will sometimes identify V with V^{**} without mentioning Ψ explicitly.

A point in $P(V)$ can be identified with a hyperplane in $P(V^*)$. By a hyperplane in $P(V)$ we mean a subset of the form $P(U)$ where $U \subset V$ is a hyperplane. More generally we introduce

Definition 2.14. (Annihilator)
For $U \subset V$ a subspace the **annihilator** *$U^o \subset V^*$ is defined as $U^o = \{f \in V^* | f(U) = \{0\}\}$.*

Of course $f(U)$ just means $\{f(u) | u \in U\}$. When $\dim U = 1$ we have $U \in P(V)$ and $U^o \subset V^*$ and we associate to it $P(U^o) \subset P(V^*)$. More generally we associate to $P(U) \subset P(V)$ the subspace $P(U^o) \subset P(V^*)$. The annihilator changes the direction of inclusions and interchanges the notions of span with intersection of subspaces. With the provision that $V \cong V^{**}$ taking the annihilator twice also brings us back. We list these properties in the following lemma and turn to its geometric consequences after.

Lemma 2.15. (Properties of the annihilator)
Suppose $U, W \subset V$ are linear subspaces.

1. *If $U \subset W$ then $W^o \subset U^o$.*

2. *$(U \cap W)^o = U^o + W^o$*

3. *$(U + W)^o = U^o \cap W^o$*

4. $\dim U^o + \dim U = \dim V$
5. $(U^o)^o = \Psi(U)$

Proof. For part 4) Choose a basis $b_1, \dots b_k$ of U and extend it to a basis of V by adding vectors $b_{k+1}, \dots b_n$ using the basis extension lemma. If we denote the dual basis by β^i then we have $\beta^i(U) = \{0\}$ if and only if $i > k$. Therefore $\dim U^o = \dim V - \dim U$.

For part 2) Choose a basis $b_1, \dots b_d$ of $U \cap W$ and extend it to a basis of U by adding vectors $b_{d+1}, \dots b_k$. Since the newly added vectors are not in W we can add additional vectors $b_{k+1}, \dots b_{k+s}$ so that $b_1, \dots b_d, b_{k+1}, \dots b_{k+s}$ form a basis of W and the $b_1, \dots b_{k+s}$ form a basis for the span $U + W$. Finally extend to a basis of V by adding some more basis vectors $b_{k+s+1}, \dots b_n$. In terms of the dual basis $\beta^1 \dots \beta^n$ we can describe the annihilators easily: First U^o is spanned by $\beta^{k+1}, \dots \beta^n$ while V^o is spanned by $\beta^{d+1} \dots \beta^k$ and $\beta^{k+s+1}, \dots \beta^n$. Together $U^o + W^o$ are spanned by $\beta^{d+1}, \dots \beta^n$. The same is true for $(U \cap V)^o$ so we established property 2).

The proof of property 3) is similar to property 2).

Property 5) is proven as follows. First $\Psi(U) \subset (U^o)^o$ because for $f \in U^o$ we have $\Psi(u)(f) = 0$ by definition. Since Ψ is an isomorphism property 4) tells us that both sides of 5) have the same dimension so the inclusion shows they must be equal. □

Writing a P in front of each of the statements gives a projective geometry equivalent and we can summarize the situation as follows.

Theorem 2.16. (Projective duality)
There is a bijection $\mathcal{D}$ from the set of projective subspaces of $P(V)$ to those of $P(V^*)$ defined by $\mathcal{D}(P(U)) = P(U^o)$. Then $\mathcal{D}$ reverses inclusions, interchanges span and intersection, turns dimension into codimension and applying it twice gives the identity.

For example in the projective plane $\dim V = 3$ the duality interchanges points with lines. For example a projective line $P(U)$ is associated to a projective point $\mathcal{D}(P(U)) = P(U^o) \subset P(V^*)$. This is indeed a point since the dimensions of U and U^o add up to 3. Under this correspondence two projective lines $P(U), P(W)$ intersecting in point $P(U \cap W)$ become two points $P(U^o)$ and $P(W^o)$ defining a projective line $P(U^o + W^o)$ in $P(V^*)$.

For example if $P(V)$ is a projective plane then a hyperplane is a line so $P(V^*)$ corresponds to the lines in $P(V)$. The space of lines through a point $X \in P(V)$ is precisely the a line X^o in $P(V^*)$. Any two points $X, Y \in P(V)$ define a unique line in $P(V)$ passing through both. From the dual this looks like two lines X^o, Y^o in $P(V^*)$ define a unique point of intersection.

Any result in projective geometry of $P(V)$ can be applied to $P(V^*)$ and then translated back to $P(V)$ to give a dual result.

For example the dual to Desargues theorem in the projective plane is the following:

Theorem 2.17. (Dual plane Desargues)
In the projective plane, imagine six distinct projective lines $A_i, B_i \subset P(V)$ where $i \in \{0, 1, 2\}$ such that the three intersection points $A_i \cap B_i$ are distinct and contained in projective line $X \subset P(V)$. Then the three projective lines $(A_i \cap A_j)(B_i \cap B_j)$ where $i \neq j \in \{0, 1, 2\}$ intersect in a single point.

Taken in projective 3-space Desargues also has a dual that looks a little different as in this case the dual to a point is not a line but a projective plane.

Theorem 2.18. (Dual 3D Desargues)
In projective three-dimensional space, imagine six distinct projective planes $A_i, B_i \subset P(V)$ where $i \in \{0, 1, 2\}$ such that the three intersection plines $A_i \cap B_i$ are distinct and contained in projective plane $X \subset P(V)$. For $i \neq j \in \{0, 1, 2\}$ denote by C_{ij} the projective plane containing the plines $(A_i \cap A_j)$ and $(B_i \cap B_j)$. The three planes C_{ij} where intersect in a single pline.

Exercises

1. (a)

Chapter 3

Riemannian geometry

The main question we want to address is: how to find the shortest path between two points in a curved space?

3.1 Two examples of non-Euclidean geometry

Before coming up with a general definition of a curved space we first consider two important examples of such Non-Euclidean geometries. First we briefly study the geometry of the sphere. Next we study the hyperbolic plane. This section is meant to illustrate some concepts we will treat more thoroughly later on.

3.1.1 Spherical geometry

On the sphere $S^2 \subset \mathbb{R}^3$ consider two points $A \neq B$. What is the shortest, straightest curve on S^2 that connects them? Notice that A, B span a linear subspace M and reflection in this mirror M preserves S^2. It seems plausible that the optimal path connecting A to B should be unique. This means it must remain unchanged under reflection in M. Hence it should be contained in the intersection of M with S^2. Such intersections are known as great circles. We will later prove that these are indeed the geodesics on the sphere.

Spherical angles, spherical triangles and spherical area. Tilings of the sphere by regular pentagons. Spherical isometries are O(3).

3.1.2 Hyperbolic geometry

We now introduce a more abstract example of a curved space called the hyperbolic plane. As a set we take it to be the open unit disk in the plane. The geodesics are said to be the Euclidean circles orthogonal to the boundary. Hyperbolic angles are Euclidean angles. Hyperbolic isometries are inversions in circles orthogonal to the boundary.

Tilings of hyperbolic space by regular k-gons for any $k \geq 7$.

Exercises

1. (a)

3.2 Riemannian charts and metrics

Definition 3.1. (Riemannian chart)
An n-dimensional **Riemannian chart** (P, g) *is an open subset $P \subset \mathbb{R}^n$ together with a* **Riemannian metric** g. *The Riemanian metric is a choice of inner product $g(p)$ for each $p \in P$ in such a way that the functions $g_{ij} : P \to \mathbb{R}$ defined by $g_{ij} = g(p)(e_i, e_j)$ are C^1.*

Perhaps the simplest Riemannian metric is to take the standard inner product of $\mathbb{R}^n$ at every point, we call this metric g_E. We can do this on any open subset $P \subset \mathbb{R}^n$ and get a Riemannian chart that we call (P, g_E). The Riemannian metric g_E is defined by $g_E(p)(v, w) = \langle v, w \rangle$ for all $p \in P$ and $v, w \in \mathbb{R}^n$. The functions $(g_E)_{ij}(p) = \langle e_i, e_j \rangle = \delta_{ij}$ are constant and hence C^1 functions.

We already met one instance of the hyperbolic plane but here is another view on the same space where we take a half plane instead of a disk. Define a Riemannian chart $(\mathbb{H}^n, g_{hyp})$ by $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ with metric $g_{hyp}(x, y)(v, w) = \frac{1}{y^2}\langle v, w \rangle$. It is not hard to check that g_{hyp} is indeed a Riemannian metric because $(g_{hyp})_{ij}(x, y) = \frac{\delta_{ij}}{y^2}$ is C^1.

Closer to every-day life we can describe the shape of an object in $\mathbb{R}^m$ by parametrizing it by a function $P \xrightarrow{\varphi} \mathbb{R}^m$ for some open $P \subset \mathbb{R}^n$. Provided φ is differentiable we can use the derivative of φ to introduce a Riemannian metric on P making it a Riemannian chart. When φ parametrizes the earth we are making a chart or map of the earth (hence the name geometry).

Lemma 3.2. (Pull-back metric)
Given a C^2 differentiable map $P \xrightarrow{\varphi} \mathbb{R}^m$, where $P \subset \mathbb{R}^m$ is open. Provided for any $p \in P$ the vectors $\partial_i\varphi(p)$ are linearly independent the following defines a Riemannian metric g_φ on P:

$$(g_\varphi)_{ij}(p) = \langle \partial_i\varphi(p), \partial_j\varphi(p)\rangle$$

Lengths and angles make sense in any Riemannian chart (P, g):

Definition 3.3. (Length, angle and volume)
Suppose (P, g) is an n-dimensional Riemannian chart. The **length** *$L(\gamma)$ of curve $\gamma : [a, b] \to P$ is the integral $L(\gamma) = \int_a^b \sqrt{g(\gamma(t))(\gamma'(t), \gamma'(t))}dt$.*
If β is another curve and $\beta(q) = \gamma(q) = p \in P$ the **angle** *between β, γ at p is the Euclidean angle between the vectors $\dot\beta(q), \dot\gamma(q)$ in the Euclidean space $(\mathbb{R}^n, g(p))$ with the standard orientation of $\mathbb{R}^n$.*
The volume $\mathrm{Vol}(M)$ of a subset $M \subset P$ in an n-dimensional Riemannian chart is defined by

$$\mathrm{Vol}(M) = \int_M \sqrt{|\det g|}dx_1 \dots dx_n$$

Here $\det g$ refers to the determinant of the matrix whose i, j-th entry is g_{ij}.

UPDATE EXAMPLE: For example we may take the sphere and geographic coordinates $G = (0, \pi) \times (-\pi, \pi)$ and $G \ni (\mu, \lambda) \overset{\varphi}{\mapsto} (\cos\lambda \sin\mu, \sin\lambda \sin\mu, \cos\mu) \in \mathbb{R}^3$. Here μ is the latitude coordinate and λ the longitude, for example Groningen is the point $\varphi(52.1601, 4.4970)\frac{\pi}{180}$ written in degrees. Explicitly the inner product $\varphi^* g_{Eucl}$ is given by calculating it at every point for the basis vectors e_1, e_2. Since the matrix for $\varphi'(p)$ is $\begin{pmatrix} \cos\lambda\cos\mu & -\sin\lambda\sin\mu \\ \sin\lambda\cos\mu & \cos\lambda\sin\mu \\ -\sin\mu & 0 \end{pmatrix}$ we get $\varphi^* g_{Eucl}(p)(e_1, e_1) = 1$, $\varphi^* g_{Eucl}(p)(e_1, e_2) = 0$, $\varphi^* g_{Eucl}(p)(e_2, e_2) = \sin^2\mu$.

For example in the hyperbolic plane the length of the vertical line between $(0, a)$ and $(0, b)$ given by the curve γ defined by $\gamma(t) = (t(b - a) + a)e_2$ with $a < b$ is given by $L(\gamma) = \int_0^1 \frac{b-a}{t(b-a)+a}dt = \log b - \log a$. Also notice that angles in the hyperbolic plane are just the Euclidean angles.

Definition 3.4. (Isometry)
An isometry between two Riemannian charts (P, g) and (Q, h) is a C^2 bijection $F : P \to Q$whose inverse is also C^2 satisfying $h(F(p))(F(v), F(w)) = g(p)(v, w)$ for any $p \in P$ and $v, w \in \mathbb{R}^n$. Here n is the dimension of P.

Since all geometric properties derive from the metric, isometries can be thought of as those transformations that preserve shape. As the name suggests the Euclidean linear isometries $O(E)$ from a Euclidean vector space to itself are examples of the above Riemannian notion of isometry. In this case we have $(P, g) = (Q, h) = (\mathbb{R}^n, g_E)$, the standard inner product in every point.

Other examples of isometries arise when we describe the same object in Euclidean space using two different Riemannian charts. For example let us make a completely different chart describing (part of) the sphere in $\mathbb{R}^3$. The stereographic map $\sigma : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ is defined as $\sigma(p) = \frac{(2p, |p|^2-1)}{|p|^2+1}$ where $p = (p_1, p_2)$. It has inverse $\sigma^{-1}(x, z) = \frac{x}{1-z}$ where $x = (x_1, x_2) \in \mathbb{R}^2$ defined on all of S^2 exept the north pole where $z = 1$.

The stereographic map σ has injective derivative at every point and is a C^2 function. Therefore we can use it to turn $P = \mathbb{R}^2$ into a Riemannian chart (P, g) with pull-back metric $g = \sigma^* g_E$. Recall we also had the geographic Riemannian chart $(G, \varphi^* g_E)$ of the S^2. The map $\sigma^{-1} \circ \varphi$ when restricted to the correct domain is an example of an isometry between Riemannian charts.

Another simple example is an isometry of $t : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $t(x, y) = (x + 1, y)$. Since $t'(p)$ is the identity at any point p we get $t^* g_{hyp} = g_{hyp}$. There are many more isometries of $\mathbb{H}^2$ and we will get back to them later.

For example the (Poincaré) disk model $\mathbb{D} = (\{u \in \mathbb{C} : |u| < 1\})$ with metric $g(u) = \left(\frac{2}{1-|u|^2}\right)^2 g_{Eucl}(u)$. Here as usual we identify $\mathbb{R}^2$ and $\mathbb{C}$. Now we claim that $\mathbb{D} \xrightarrow{\phi} \mathbb{H}$ given by $\phi(u) = \frac{u+i}{iu+1}$ is an isometry. Here we also identified the upper half plane $\mathbb{H}$ with a subset of $\mathbb{C}$.

To verify this we compute using complex numbers as much as possible to avoid lengthly expressions. First we use the fact that for a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ we may identify the derivative (a linear transformation in $L(\mathbb{R}^2, \mathbb{R}^2)$) with multiplication by $f'(z)$. Also identifying a vector v by $a + ib$ the Euclidean inner product becomes $\langle v, w\rangle = \mathrm{Re} v\bar{w}$. The metric on $\mathbb{H}$ is then written as $g_{hyp}(z)(v, w) = \frac{\mathrm{Re} v\bar{w}}{(\mathrm{Im}(z))^2}$. The pull back of this metric along ϕ becomes $\phi^* g_{hyp}(u)(v,w) = \mathrm{Re}\frac{\phi'(u)v\overline{\phi'(u)w}}{(\mathrm{Im}(\phi(u)))^2} = \frac{|\phi'(u)|^2}{(\mathrm{Im}(\phi(u)))^2}\mathrm{Re}(v\bar{w}) = \frac{|\phi'(u)|^2}{(\mathrm{Im}(\phi(u)))^2} g_{Eucl}(u)$. Finally $|\phi'(u)|^2 = \frac{4}{|iu+1|^4}$ and $\mathrm{Im}\phi(u) = \frac{1-|u|^2}{|iu+1|^2}$ finishing the computation.

For example, one can study the hyperbolic plane using a hyperboloid $\mathcal{H} = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2-z^2 = -1\}$. Pulling back g_{Mink} on $\mathbb{R}^3$ along the inclusion of $\mathcal{H} \stackrel{\iota}{\to} \mathbb{R}^3$ gives a metric on $\mathcal{H}$ that makes it isometric to $\mathbb{H}$. In other words, there is a metric preserving diffeomorphism between $\mathbb{H}$ and $\mathcal{H}$.

A similar computation shows that isometries of the hyperbolic plane $\mathbb{H}$ are given by the linear fractional tranformations $z \mapsto \frac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$ and $ad-bc = 1$. In fact these are all orientation preserving isometries but we will not show this here.

Definition 3.5. *(Volume)*

Exercises

1. (a)

3.3 Calculus of variations

The purpose of this section is to find a differential equation for the shortest curve between two points in a Riemannian chart (P,g), where $P \subset \mathbb{R}^n$. Since the length of a curve γ is given as $\int_0^1 |\dot\gamma(t)|dt$ we are trying to minimize the function Length on some space of curves in $\mathbb{R}^n$. This is reminiscent of minimizing ordinary functions by setting their derivative to zero and is known as calculus of variations. It makes sense to generalize a little and treat more general functions S on the space of curves.

Consider the space $\mathcal{C}$ of C^2 curves $\gamma : [0,1] \to \mathbb{R}^n$. For a C^2 function $\mathcal{L} : \mathbb{R}^{2n+1} \to \mathbb{R}$ define $S : \mathcal{C} \to \mathbb{R}$ by $S(\gamma) = \int_0^1 \mathcal{L}(\gamma(t), \gamma'(t), t)dt$.

Problem: Suppose S obtains a minimum value, how do we find it?

Theorem 3.6. *(Euler-Lagrange equation)*
If $\gamma \in \mathcal{C}$ is a minimum of S then for all $i \in \{1,\dots n\}$:

$$\partial_i \mathcal{L}(\gamma(t), \gamma'(t), t) = \frac{d}{dt}(\partial_{n+i}\mathcal{L}(\gamma(t), \gamma'(t), t))$$

The proof of this theorem depends on a simple lemma

Lemma 3.7. *Suppose M is a C^2 function $M : [0,1] \to \mathbb{R}^n$. If M is such that for all C^2 functions $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1) = 0$ we have*

$$\int_0^1 M(t)h(t)dt = 0$$

then $M(t) = 0$ for all $t \in [0,1]$.

Proof. It suffices to treat the case $n = 1$. The general case follows by restricting to one of the coordinates.

Suppose $M(z) \neq 0$ for some $z \in [0,1]$, say $M(z) > 0$. By continuity of M there must exist a small interval $[z_1, z_2] \subset [0,1]$ where $M > 0$. For a contradiction define the function $h : [0,1] \to \mathbb{R}$ by $h(t) = -(t-z_1)^3(t-z_2)^3$ if $t \in [z_1, z_2]$ and $h(t) = 0$ outside this interval. We leave it to the reader to check that h is C^2. Then Mh is positive on (z_1, z_2) and equal to 0 outside this interval so $0 = \int_0^1 M(t)h(t)dt = \int_{z_1}^{z_2} M(t)h(t)dt > 0$. □

Proof. (Of Theorem 3.6)
Let us assume curve $\gamma \in \mathcal{C}$ minimizes S in the sense that for any other curve $\beta \in \mathcal{C}$ $S(\gamma) \leq S(\beta)$.

Choose $i \in \{1,\dots,n\}$ and consider perturbing the curve γ by adding an arbitrary function h in the i direction. More precisely choose some fixed C^2 function $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1) = 0$. The function $V : (-v,v) \times [0,1] \to P$ defined by $V(\epsilon, t) = \gamma(t) + \epsilon h(t)e_i$ represents a family of curves, one for each value of ϵ and v is chosen so that all the curves actually have image contained in P. So for any fixed $|\epsilon| < v$ we get a curve $V_\epsilon : [0,1] \to P$ defined by $V_\epsilon(t) = V(\epsilon, t)$. For example $V_0 = \gamma$ and for small ϵ we get a curve V_ϵ deviating slightly from γ.

By assumption $S(V_\epsilon) \geq S(V_0)$ and this means that the function $H : (-v,v) \to \mathbb{R}$ defined by $H(\epsilon) = S(V_\epsilon)$ has a minimum at $\epsilon = 0$. Ordinary calculus tells us that $H'(0) = 0$. Unpacking the definition of H this means

$$0 = H'(0) = \frac{d}{d\epsilon}|_{\epsilon=0} \int_0^1 \mathcal{L}(V(\epsilon,t), \partial_t V(\epsilon,t), t)dt = \int_0^1 \partial_\epsilon|_{\epsilon=0} \mathcal{L}(V(\epsilon,t), \partial_t V(\epsilon,t), t)dt \tag{3.1}$$

To apply the chain rule in a clean way we introduce $W : (-v, v) \times \mathbb{R} \to P \times \mathbb{R}^{n+1}$ defined by $W(\epsilon, t) = (V(\epsilon, t), V'(\epsilon, t), t)$. Then the derivative with respect to ϵ inside the integral is equal to $(\mathcal{L} \circ W)'(0, t)e_1 = \mathcal{L}'(W(0, t))W'(0, t)e_1$ by the chain rule[1]. Now $W'(0, t)e_1 = h(t)e_i + h'(t)e_{n+i}$ so our integrand from (3.1) is

$$(\mathcal{L} \circ W)'(0, t)e_1 = \mathcal{L}'(W(0, t))(h(t)e_i + h'(t)e_{n+i}) = (\partial_i \mathcal{L})(\gamma(t), \gamma'(t), t)h(t) + (\partial_{i+n}\mathcal{L})(\gamma(t), \gamma'(t), t)h'(t)$$

Partial integration gets rid of the h' in the second term (using $h(0) = h(1) = 0$) and turns our integral into

$$0 = H'(0) = \int_0^1 \partial_i \mathcal{L}(\gamma(t), \gamma'(t), t)h(t) - \partial_t\big((\partial_{i+n}\mathcal{L})(\gamma(t), \gamma'(t), t)\big)h(t)dt = \int_0^1 M(t)h(t)dt = 0$$

$$M = \partial_i \mathcal{L}(\gamma(t), \gamma'(t), t) - \partial_t\big((\partial_{i+n}\mathcal{L})(\gamma(t), \gamma'(t), t)\big)$$

Since this must hold for arbitrary choices of h Lemma 3.7 shows $M = 0$ finishing the proof. □

Exercises

1. (a)

3.4 Geodesics

Our main application of the calculus of variations is to minimize the length of curves in a Riemannian chart. Such curves will be shown to be solutions to a system of ordinary differential equations known as the geodesic equation (3.4).

Definition 3.8. *Any C^2 curve in a Riemannian chart that solves the geodesic equation* (3.4) *is called a* **geodesic** *(curve).*

For technical reasons it is easier not to directly find curves that minimize the length but rather minimize the energy $S(\gamma) = \int_0^1 |\gamma'(t)|^2 dt$. By reparametrizing we can assume our curve γ minimizing the length has unit speed, so $|\dot{\gamma}(t)| = 1$. Since $1^2 = 1$ we see γ must also minimize the easier S which uses the squared norm.

For example take a curve in flat 2-space $\gamma : [0, 1] \to \mathbb{R}^2$. We aim to minimize $S(\gamma) = \int_0^1 |\gamma'(t)|^2 dt$ so $\mathcal{L}(x, y, t) = y_1^2 + y_2^2$.

The Euler-Lagrange equations we need to solve are: $\gamma''(t) = 0$ since

$$\frac{d}{dt}(\partial_{2+i}\mathcal{L}(\gamma(t), \gamma'(t), t) = \frac{d}{dt}(2\gamma_i'(t)) = 2\gamma_i''(t) = 0 = \partial_i \mathcal{L}(\gamma(t), \gamma'(t), t)$$

It follows that $\gamma(t) = at + b$ for some $a, b \in \mathbb{R}^2$.

Moving on to the general case, consider a curve $\gamma : [0, 1] \to \mathbb{R}^n$. For the argument it is important to allow all curves, not just those in the Riemannian chart (P, g) with $P \subset \mathbb{R}^n$. Writing $\gamma(t) = \sum_{i=1}^n \gamma_i e_i$ we find

$$|\gamma'(t)|^2 = \sum_{i,j} g(\gamma(t))(\gamma_i'(t)e_i, \gamma_j'(t)e_j) = \sum_{i,j} g_{ij}(\gamma(t))\gamma_i'(t)\gamma_j'(t)$$

In this case we seek to minimize $S(\gamma) = \int_0^1 |\gamma'(t)|^2 dt = \int_0^1 \sum_{i,j} g_{ij}(\gamma(t))\gamma_i'(t)\gamma_j'(t)dt$ so $\mathcal{L}(x, y, t) = \sum_{i,j=1}^n g_{ij}(x)y_i y_j$ is a function from $\mathbb{R}^{2n+1}$ to $\mathbb{R}$.

The Euler-Lagrange equations now read for any fixed $k \in \{1, \dots n\}$:

$$\partial_k \mathcal{L}(\gamma(t), \gamma'(t), t) = \frac{d}{dt}(\partial_{n+k}\mathcal{L}(\gamma(t), \gamma'(t), t)) \tag{3.2}$$

The left hand side is $\sum_{i,j=1}^n (\frac{\partial g_{ij}}{\partial x_k}) y_i y_j$. For the right hand side we first compute

$$\partial_{n+k}\mathcal{L}(x, y, t)) = \sum_{j=1}^n g_{kj}(x)y_j + \sum_{i=1}^n g_{ik}(x)y_i = 2\sum_{i=1}^n g_{ik}(x)y_i$$

so the right hand side of (3.2) is

$$\frac{d}{dt}(\partial_{n+k}\mathcal{L}(x, y, t))) = 2\frac{d}{dt}\sum_{i=1}^n g_{ik}(\gamma(t))\gamma_i'(t) = 2\sum_{i,j} \frac{\partial g_{ik}}{\partial x_j}(\gamma(t))\gamma_i'(t)\gamma_j'(t) + 2\sum_{i=1}^n g_{ik}(\gamma(t))\gamma_i''(t)$$

[1] Here e_1 refers to the first coordinate in $(-v, v) \times [0, 1]$, which is ϵ. Also $W'(0, t)e_1 = \partial_\epsilon W(\epsilon, t)|_{\epsilon=0}$.

Using symmetry to interchange the dummy indices i, j we get $\sum_{i,j} \frac{\partial g_{ik}}{\partial x_j}(\gamma(t))\gamma_i'(t)\gamma_j'(t) = \sum_{i,j} \frac{\partial g_{jk}}{\partial x_i}(\gamma(t))\gamma_i'(t)\gamma_j'(t)$

Putting it all together we found a differential equation for γ:

$$0 = 2\sum_{s=1}^{n} g_{ks}(\gamma(t))\gamma_s''(t) + \sum_{i,j=1}^{n}\Big(\frac{\partial g_{jk}}{\partial x_i}(\gamma(t)) + \frac{\partial g_{ik}}{\partial x_j}(\gamma(t)) - \frac{\partial g_{ij}}{\partial x_k}(\gamma(t))\Big)\gamma_i'(t)\gamma_j'(t) \tag{3.3}$$

It is convenient to isolate the second derivative term $\gamma_s''(t)$. This can be achieved since the matrix $(g_{ij}(\gamma(t)))$ describing the metric is positive definite and symmetric. It follows that it has only positive eigenvalues and hence positive determinant. To see this pick an eigenvector v with eigenvalue λ and write $v^T(g_{ij}(\gamma(t)))v = v^T\lambda v = \lambda|v|^2 > 0$. It thus has an inverse $(g_{rk}^{-1}(\gamma(t)))$ so $\sum_k g_{rk}^{-1}(\gamma(t))g_{ks}(\gamma(t)) = \delta_{rs}$ (Kronecker delta). Applying this to both sides of our differential equations we get the **geodesic equation**. For fixed $r \in \{1,\dots n\}$ we have:

$$0 = \gamma_r''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^r(\gamma(t))\gamma_i'(t)\gamma_j'(t) \tag{3.4}$$

For any value of $i, j, r \in \{1, \dots n\}$ the function $\Gamma_{ij}^r : P \to \mathbb{R}$ is known as the **Christoffel symbols** and is defined as:

$$\Gamma_{ij}^r(p) = \frac{1}{2}\sum_{k=1}^{n} g_{rk}^{-1}(p)\Big(\frac{\partial g_{jk}}{\partial x_i}(p) + \frac{\partial g_{ik}}{\partial x_j}(p) - \frac{\partial g_{ij}}{\partial x_k}(p)\Big) \tag{3.5}$$

For example the eight Christoffel symbols for the sphere with geographic coordinates are $\Gamma_{ij}^1 = 0$ except $\Gamma_{22}^1 = -\cos\mu\sin\mu$. $\Gamma_{11}^2 = \Gamma_{22}^2 = 0$ and $\Gamma_{12}^2 = \Gamma_{21}^2 = \cot\mu = \frac{\cos\mu}{\sin\mu}$. So the geodesic equations for $\gamma = (\gamma_1, \gamma_2)$ is $\ddot{\gamma}_1 - \cos\mu\sin\mu\dot{\gamma}_2\dot{\gamma}_2 = 0 = \ddot{\gamma}_2 + 2\cot\mu\dot{\gamma}_1\dot{\gamma}_2$. While a little complicated at first sight it should be clear that the meridians $\gamma(t) = (t, c)$ for some constant $c \in \mathbb{R}$ satisfy this equation.

Returning to the general case: the fundamental theorem on the existence and uniqueness of ordinary differential equations assures us that geodesics always exist. Intuitively the next theorem states that in a Riemannian chart one can start walking 'straight' in any direction at any point of the chart.

Theorem 3.9. (Existence and uniqueness of geodesics)
In a Riemannian chart (P, g) any $p \in P \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$ determines a unique C^2 curve $\gamma : [C, D] \to P$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ and for all m we have $\gamma_m''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^m \gamma_i'(t)\gamma_j'(t)$. The domain $[-D, D]$ is not quite unique but if we have another curve $\tilde{\gamma}$ with the same properties and domain $[A, B]$ then $\gamma = \tilde{\gamma}$ on $[A, B] \cap [C, D]$.

We arrived at geodesics by attempting to minimize the distance between two points but what we found is actually more like the curves that are 'straight'. Depending on global issues travelling a straight line may or may not actually minimize the distance travelled.

Exercises

1. (a)