Geometry Tutorial 9

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1. For an open set $P \subset \mathbb{R}^6$ defined by $x_1 \neq 0$ define the function $f : P \overset{f}{\mapsto} \mathbb{R}^4$ by $f(x) = (\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}, \frac{x_5}{x_1})$ Is f injective? What about $f'(1,1,1,1,1,1) : \mathbb{R}^6 \to \mathbb{R}^4$?

   Solution: f is not injective since if $f(x) = f(y)$ then $x_i = y_i$ for $i = 1, \ldots, 5$ but $x_6 \neq y_6$ necessarily.

   We compute $f'(x)$ to be:

$$f'(x) = \begin{bmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} & 0 & 0 & 0 & 0 \\ -\frac{x_3}{x_1^2} & 0 & \frac{1}{x_1} & 0 & 0 & 0 \\ -\frac{x_4}{x_1^2} & 0 & 0 & \frac{1}{x_1} & 0 & 0 \\ -\frac{x_5}{x_1^2} & 0 & 0 & 0 & \frac{1}{x_1} & 0 \end{bmatrix}.$$

   Since $\frac{1}{x_1} \neq 0$ for any $x \in P$, this matrix always has rank 4 for any $x \in P$. By the rank-nullity theorem we have that $n = \text{rank}(f'(x)) + \dim(\ker(f'(x)))$. Since $n = 6$, we have that the dimension of the kernel must be 2 and therefore the kernel cannot be trivial and therefore f' is not injective.

2. Pull back the standard Euclidean (Riemannian) metric on $\mathbb{R}^3$ using $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(x,y) = (x, y, |(x,y)|^2)$.

   (a) Does this define a Riemann metric on $\mathbb{R}^2$?

   Solution: We have the map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ that maps $(x,y) \mapsto (x, y, x^2 + y^2)$. First, we need to show that ϕ' is injective. We compute the derivative of this map to be:

$$\phi'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2y \end{bmatrix}.$$

   Using the rank-nullity theorem we see that the $\dim(\ker(\phi')) = 0$ and hence ϕ' is injective.

Next we check whether $\phi^* g_{ij}(x,y) = \phi^* g(x,y)(e_i, e_j)$, where e_i and e_j are basis vectors, are C^1. We compute:

$$\begin{aligned}
\phi^* g_{11}(x,y) &= g(\phi(x,y))(\phi'(x,y)e_1, \phi'(x,y)e_1) \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \right\rangle \\
&= 4x^2 + 1.
\end{aligned}$$

$$\begin{aligned}
\phi^* g_{12}(x,y) &= g(\phi(x,y))(\phi'(x,y)e_1, \phi'(x,y)e_2) \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} \right\rangle \\
&= 4xy.
\end{aligned}$$

$$\begin{aligned}
\phi^* g_{22}(x,y) &= g(\phi(x,y))(\phi'(x,y)e_1, \phi'(x,y)e_1) \\
&= \left\langle \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} \right\rangle \\
&= 4y^2 + 1.
\end{aligned}$$

Where we note that $4x^2 + 1, 4xy, 4y^2 + 1$ are C^1.

Lastly, we check whether $\phi^* g$ does define an inner product. Is is bilinear, symmetric and positive definite because $g(\phi(x,y))$ is bilinear, symmetric and positive definite. We note that $\phi^* g(x,y)(v,v) = g(\phi(x,y))(\phi'(x,y)v, \phi'(x,y)v) = 0$ if and only if $\phi'(x,y)v = 0$ which is the case if and only if $v = 0$. Proving that $\phi^* g$ is an inner product. This proves that $\phi^* g$ defines a Riemannian metric.

(b) Compute the circumference of the circle with radius $r > 0$ in the Riemannian chart $(\mathbb{R}^2 - \{0\}, \varphi^* g_E)$.

Solution: We have that a circle of radius r has parametrization $\gamma_r(t) = (r\cos(t), r\sin(t))$ for $t \in (0, 2\pi)$. The length of the curve is given by $L(\gamma_r) = \int_0^{2\pi} |\dot{\gamma_r}(t)| dt$. Computing the derivative of the curve we get $\dot{\gamma_r}(t) = (-r\sin(t), r\cos(t))$.

Note that $\phi'(\gamma_r(t))\dot{\gamma_r}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ r\cos(t) & r\sin(t) \end{pmatrix} \begin{pmatrix} -r\sin(t) \\ r\cos(t) \end{pmatrix} = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \\ 0 \end{pmatrix}$.

Hene the square norm is given by:

$$\begin{aligned}|\dot{\gamma_r}(t)|^2 &= \phi^* g(\dot{\gamma_r}, \dot{\gamma_r}) \\ &= g(\phi(\gamma_r(t)))(\phi'(\gamma_r(t))\dot{\gamma_r}(t), \phi'(\gamma_r(t))\dot{\gamma_r}(t)) \\ &= \left\langle \begin{pmatrix} -r\sin(t) \\ r\cos(t) \\ 0 \end{pmatrix}, \begin{pmatrix} -r\sin(t) \\ r\cos(t) \\ 0 \end{pmatrix} \right\rangle \\ &= r^2\end{aligned}$$

We compute the length of the curve to be:

$$L(\gamma_r) = \int_0^{2\pi} r dt = 2\pi r.$$

(c) How does your answer compare to the usual $2\pi r$?

Solution: The radius of these circles are equal to a circle of radius r in the plane.

3. Consider the hyperbolic half-plane $\mathbb{H}^2$ with metric $g(p)(v, w) = \frac{1}{y^2}\langle v, w\rangle_E$.

(a) Compute the length of the straight line connecting $A(0, 1)$ and $B(2, 1)$.

Solution: We can parametrize by $\gamma(t) = (t, 1)$, $t \in (0, 2)$. Hence, $\dot{\gamma}(t) = (1, 0) = e_1$ for all t. Hence,

$$\|\dot{\gamma}(t)\|^2 = (\gamma(t))(\dot{\gamma}(t), \dot{\gamma}(t)) = \frac{1}{1^2}\langle e_1, e_1\rangle = 1, \quad \forall t,$$

and the length becomes

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt = \int_0^1 \sqrt{1} dt = 2.$$

(b) Consider the circle with center $M(1, 0)$ passing through A and B. Calculate the length of the arc between A and B. Compare your result with part (a). *Hint:* Use software to evaluate the integral.

Solution: In this case, we can parametrize by $\gamma(t) = (1 + \sqrt{2}\cos t, \sqrt{2}\sin t)$ with $t \in (\pi/4, 3\pi/4)$. (Draw a picture!) Hence, $\dot{\gamma}(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t)$, and we get

$$L(\gamma) = \int_{\pi/4}^{3\pi/4} \sqrt{\frac{1}{2\sin^2 t} \cdot (2\sin^2 t + 2\cos^2 t)} dt = \int_{\pi/4}^{3\pi/4} \frac{1}{\sin t} dt.$$

If you're the kind of person who enjoys solving integrals, you can solve it exactly to obtain $2\coth^{-1}(\sqrt{2})$. Anyway, the answer is approximately 1.76, which is shorter than the "straight" line from part (a).