Geometry Tutorial 8

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1. Look up a discussion of the Banach-Tarski paradox to convince yourself that defining area on the sphere is a subtle issue.

> **Solution:** "Given a solid ball in 3dimensional [Euclidean] space, there exists a [partition] of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Indeed, the reassembly process involves only orientation preserving isometries [rigid motions]." Source: Wikipedia
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> This paradox implies that there is no definite area function ($P(B^3)$ denotes the set of all subsets of the ball) $A : P(B^3) \to [0, \infty)$ which has all the properties we would like such a function to have, e.g. invariance under isometries. The solution to this paradox is to drop the requirement that area should be defined on all subsets of B^3. Instead, we restrict the domain of A to some suitable "sigma algebra" $\mathcal{A}$ of subsets of B^3, making sure that $\mathcal{A}$ still includes all "reasonable" subsets, such as solid angles subtended by spherical polygons.
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> Remark: There is already a similar problem with defining the "length" of arbitrary subsets $A \subseteq \mathbb{R}$. It turns out to be impossible to define a "measure" $\mu : \mathbb{R} \to [0, \infty]$ which is additive, translation invariant, and which has the correct value $b - a$ on intervals.

2. Use integration to calculate the surface area of a unit sphere. Do the same for a time zone of angle α.

> **Solution:** The parametrization of a sphere with radius ρ in polar coordinates is given by: $(r(u, v) = (\rho \sin u \cos v, \rho \sin u \sin v, \rho \cos u)$.
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> The area can be computed by $\int_0^{2\pi} \int_0^{\pi} \| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \| dudv$. Computing, this we obtain $\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \| = \sin(u)$. Therefore,
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> $$\int_0^{2\pi} \int_0^{\pi} \| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \| dudv = \int_0^{2\pi} \int_0^{\pi} \sin(u) dudv = 4\pi$$

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For the surface area of a time zone of angle α we integrate from 0 to α instead of integrating over 0 to 2π. This means that integral of the surface area reduces to:

$$\int_0^\alpha \int_0^\pi \sin(u) du dv = 2\alpha.$$

3. Explain why the angle sum in a spherical triangle is always greater than π. The angle sum is also bounded from above. What's the best upper bound you can find?

Solution: We have seen that the angle sum in a spherical triangle is the area of said triangle plus π. Since the area is always positive, the angle sum is strictly greater than π.

Since a spherical triangle is defined as the intersection of three half-spaces with S^2, the area is bounded above by 2π. Hence, the angle sum is at most 3π. In fact, this is the supremum. (Convince yourself by drawing a picture.)

4. Let E be a Euclidean vector space, $v \in E$. Recall that the half-space H_v is defined as

$$H_v = \{w \in E : v \cdot w \geq 0\}.$$

Prove that H_v is convex.

Solution: Let $w_1, w_2 \in H_v$. If $w \in [w_1, w_2]$, we have for some $t \in [0, 1]$:

$$w = tw_1 + (1 - t)w_2$$

so that

$$v \cdot w = v \cdot (tw_1 + (1 - t)w_2) = t(v \cdot w_1) + (1 - t)(v \cdot w_2) \geq 0.$$

Thus, $w \in H_v$, and H_v is convex.

5. Imagine an ant living on the surface of a sphere. How could it define the distance between two points? Does this define a metric in the 'traditional' sense? Can the ant detect that it lives on a curved surface?

Solution: The shortest distance between two points on a sphere is given by the great-circle distance. These are the equivalents of straight lines on a sphere (as we will see in the second part of the course). This does give rise to a metric on the sphere. All properties except for the triangle inequality are easy to prove.

The ant could of course prove that its world is round, by going all the way around it. But that doesn't mean that the surface is curved, it could also live on a piece of paper, which is glued to make a cylinder. That would still have curvature zero. One way to test the curvature would be to measure the angle sum in a triangle. If it is greater than π, then the average curvature is positive. By letting the side lengths go to zero, it can also determine the local curvature.

6. Spatial rotations. In this exercise, E denotes a three-dimensional Euclidean vector space. All the planes and lines mentioned here pass through the origin (are linear subspaces).
   (a) Using the determinant and the reflection theorem, prove that the composition of two rotations in E is again a rotation.
   (b) Prove that if you know two angles and one side of a spherical triangle then the triangle is determined.
   (c) Imagine plane F containing line L and another line M in E. Show that there exists a rotation ρ with axis L such that $\rho(F)$ contains M.
   (d) Imagine a pair of planes F, G intersecting in line $L = F \cap G$ and prove that for any line M the rotation ρ from the previous part satisfies $s_F \circ s_G = s_{\rho(F)} \circ s_{\rho(G)}$.
   (e) Given two rotations r, r' written as $r = s_F \circ s_G$ and $r' = s_{F'} \circ s_{G'}$, show that we can find new planes A, B, C such that $r = s_A \circ s_B$ and $s_B \circ s_C$.
   (f) Conclude that the composition $r \circ r'$ equals the rotation $s_A \circ s_C$. How does this relate to the spherical triangle from part b?
   (g) How do the angles of the spherical triangle defined by planes A, B, C relate to the angles of the rotations r, r' and its composition?