Geometry Tutorial 7

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1. Figure out how many faces, edges and vertices a soccer ball has.

> **Solution:** There are 12 pentagons and around each pentagon 6 hexagons meet. Every hexagon meets 3 pentagons so we have $12 * 5/3 = 20$ hexagons. So $F = 32$. Every edge is part of two faces so looking at the faces we have $2E = 12 * 5 + 20 * 6$ meaning $E = 90$. By Euler's formula this means V must be 60.

2. Imagine a subset S of Euclidean affine 3-space built from finitely many octahedra with side length 1 as follows. Each octahedron has four triangular faces colored black and four colored white such that adjacent faces have opposite color. Two octahedra are either disjoint or intersect in a pair of black faces. Whenever two octahedra meet we delete the face along which they meet and also remove duplicate edges and vertices. Counting the number of faces F the edges E and vertices V in this way, what values can $\chi_S = V - E + F$ take?

> **Solution:** We claim that χ_S only depends on the underlying graph Γ. The graph Γ has a vertex for every octahedron and an edge between vertices if the corresponding octahedra share an edge. Moreover, $\chi_S = 2\chi_\Gamma$ where $\chi_\Gamma = V_\Gamma - E_\Gamma$. It follows from this claim that the values χ can take are precisely all even integers.
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> To prove our claims we first focus on the case where Γ is a tree. When the tree has a single vertex the $\chi_S = 2 = 2\chi_\Gamma$ by Euler's formula since the octahedron itself is a convex polyhedron. Next, adding a leaf to the tree increases V by 3, E by 9 and F by 6 since this is what happens when we glue another octahedron. This does not change χ_S and also $2\chi_\Gamma$ does not change. By induction on the number of vertices we thus know it holds for all cases where Γ is a tree.
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> Taking disjoint unions of octahedra will allow χ_S to attain any positive even number and this still agrees with χ_Γ.
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> To get negative even numbers we start adding additional edges to the tree and every edge will decrease χ_Γ by two. This corresponds to selecting two octahedra and gluing them along a pair of faces. The gluing will decrease F by 2, E by 3 and V by 3 meaning χ_S also decreases by 2.

Any connected graph is obtained from a spanning tree by adding edges so we are done.

3. Explain why the cube $[-1,1]^3 \subset \mathbb{R}^3$ has only finitely many rotational symmetries. By a rotational symmetry we mean an affine rotation that sends the cube to itself (as a set). Is the same true for any convex polyhedron?

Solution: Yes it is true for all non-degenerate convex polyhedra but not for the degenerate polyhedron that is the convex hull of a single point.

The argument is as follows. Non-degeneracy means the vertices include an affine frame so any rotational symmetry is determined uniquely by where it sends the vertices. Also any rotational symmetry must send the set of vertices to itself, i.e. permute the vertices amongst themselves. There are finitely many permutations of the vertices (it is a finite set) so there can only be finitely many rotational symmetries.

4. Write a translation in affine 3-space $\mathcal{E}$ as the composition of two reflections. If r is an affine rotation, prove that $r \circ t_u \circ r^{-1} = t_{\overrightarrow{r}(u)}$ for any vector u in the underlying vector space.

Solution: For the first part, Let u be the translation vector, and choose any point $O \in \mathcal{E}$. If $F = \{u\}^{\perp}$, we can define $\mathcal{F}$ as the (unique!) plane through O parallel to F. Likewise, if O' is determined by $\overrightarrow{OO'} = u/2$, define $\mathcal{F}'$ as the plane through O' parallel to F. It's not hard to see that $t_u = \sigma_{\mathcal{F}'} \circ \sigma_{\mathcal{F}}$.

If r has center (fixed point) $O \in \mathcal{E}$ then there is a linear rotation $\rho = \overrightarrow{r}$ such that for all points A the image $r(A)$ is determined by $\rho\overrightarrow{OA} = \overrightarrow{Or(A)}$.

Set $B = r^{-1}A$ and $C = t_u B$ and $D = rC$. By definition of affine rotation and translation we have the relations $\rho^{-1}\overrightarrow{OA} = \overrightarrow{OB}$ and $\overrightarrow{BC} = u$ and $\overrightarrow{OD} = \rho\overrightarrow{OC}$. This means that $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \rho\overrightarrow{OC} - \overrightarrow{OA} = \rho\overrightarrow{OC} - \rho\overrightarrow{OB} = \rho(\overrightarrow{OC} - \overrightarrow{OB}) = \rho\overrightarrow{BC} = \rho(u)$ in other words $r \circ t_u \circ r^{-1}A = t_{\overrightarrow{r}(u)}A$ for any $A \in \mathcal{E}$. Notice this proof works in any dimension.

5. If vector space V is a direct sum of subspaces A and B then choosing an orientation $\mathcal{O}_A$ of A and $\mathcal{O}_B$ of B will determine an orientation $\mathcal{O}$ of V by the rule that for $a \in \mathcal{O}_A$ and $b \in \mathcal{O}_B$ we want the concatenation (a,b) to be in $\mathcal{O}$. What happens to the orientation of V when we reverse the orientation of A? Does it also reverse? (By reversing we mean passing to the other one of the two possible orientations.)

Solution: Yes. In the above case, the change of basis from (a,b) to (a',b) matrix is block-diagonal, so its determinant is the product of determinants of each block. Changing the A-block by a minus sign, and leaving the B-block invariant gives an overall minus sign.