Geometry Tutorial 6

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February 27, 2020

1. Let E be some vector space of dimension n. An (ordered) basis $(e_1, \dots, e_n)$ is a linearly independent spanning set for E. Recall (or convince yourself) that if $(e'_1, \dots, e'_m)$ is another basis for E, then $n = m$.

   (a) Suppose that $(e_1, \dots, e_n)$ and $(e'_1, \dots, e'_n)$ are two ordered bases for E. Explain why there is a unique linear map $T : E \to E$ such that $T(e_i) = e'_i$ for all $i = 1, \dots, n$.

   Solution: Given $x \in E$ there are unique scalars $\lambda_i \in \mathbb{K}$ such that $x = \sum_{i=1}^n \lambda_i e_i$. The map

   $$T(x) = T\left(\sum_{i=1}^n \lambda_i e_i\right) = \sum_{i=1}^n \lambda_i e'_i$$

   clearly has the required properties. (It is well-defined, because the scalars are unique.) If $\hat{T}$ were another such map, then

   $$e'_i = T(e_i) = \hat{T}(e_i), \quad \forall i = 1, 2, \dots, n.$$

   So, by linearity

   $$T(x) = \sum_{i=1}^n \lambda_i e'_i = \sum_{i=1}^n \lambda_i \hat{T}(e_i) = \hat{T}\left(\sum_{i=1}^n \lambda_i e_i\right) = \hat{T}(x) \quad \forall x \in E,$$

   thus $T = \hat{T}$.

   (b) Why is this map bijective?

   Solution: Suppose that $T(x) = \sum_{i=1}^n \lambda_i e'_i = 0$. Then, by linear independence of the e'_i, we have that $\lambda_i = 0$ for all $i = 1, \dots, n$, so $x = 0$. This shows that $\ker(T) = \{0\}$.

   If $T(x) = T(y)$, then by linearity, $T(x - y) = 0$, so $x - y \in \ker(T) = \{0\}$. It thus follows that $x - y = 0$, so $x = y$, and T is injective.

Having observed that $\dim(\ker(T)) = 0$, it follows from the rank-nullity theorem that $\dim(\operatorname{ran}(T)) = n$. The only n-dimensional subspace of E is E itself, so $\operatorname{ran}(T) = E$, and T is surjective.

(c) We call two ordered bases *equivalent* if this unique linear map has positive determinant. Prove that this defines an equivalence relation on the set of all ordered bases.

Solution:
(Reflexivity) If the two bases are the same, then the linear map connecting them is the identity, which has determinant 1.

(Symmetry) If $(e_1, \ldots, e_n) \sim (e'_1, \ldots, e'_n)$ via the linear map T, then $(e'_1, \ldots, e'_n) \sim (e_1, \ldots, e_n)$ via T^{-1}. If T has positive determinant, then so does T^{-1}, because $\det T^{-1} = \frac{1}{\det T}$.

(Transitivity) is similar. This follows from $\det(T_1 T_2) = \det T_1 \cdot \det T_2$.

(d) Prove that this equivalence relation partitions the set of ordered bases into two equivalence classes.

Solution: Let $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ be arbitrary ordered basis for E. We claim that either $(e'_1, \ldots, e'_n) \sim (e_1, e_2, \ldots, e_n)$ or $(e'_1, \ldots, e'_n) \sim (-e_1, e_2, \ldots, e_n)$, and that $(e_1, e_2, \ldots, e_n)$ and $(-e_1, e_2, \ldots, e_n)$ are not equivalent. This shows clearly that there are two distinct equivalence classes.

To prove the claim, notice that the change of basis matrix between $(e_1, e_2, \ldots, e_n)$ and $(-e_1, e_2, \ldots, e_n)$ has determinant -1. So, if the change of basis matrix between $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ has negative determinant, then the change of basis matrix between $(e'_1, \ldots, e'_n)$ and $(-e_1, e_2, \ldots, e_n)$ has positive determinant again.

2. Let E be a Euclidean vector space of dimension n, and consider the unit ball

$$B^n = \{x \in E : \|x\| \leq 1\} \subset E.$$

Prove that B^n is convex.

Solution: Let $x, y \in B^n$, and consider the line segment

$$[x, y] = \{tx + (1-t)y : t \in [0, 1]\} \subset E$$

connecting them. For a point $z \in [x, y]$, we have

$$\|z\| = \|tx + (1-t)y\| \leq |t|\|x\| + |1-t|\|y\| \leq t + (1-t) = 1,$$

so $z \in D^n$, and we are done.

3. We aim to prove that if there are triangles $\mathbb{A} = [A_0, A_1, A_2]$ and $\mathbb{B} = [B_0, B_1, B_2]$ in the Euclidean affine plane such that $\angle A_i A_j A_k = \angle B_i B_j B_k$ for all triples i, j, k of distinct numbers between 0 and 2 then they are related by a composition of an isometry and a dilation (aka a similarity).

   (a) Prove that there is a unique translation τ sending B_0 to A_0.

   (b) Next if $u_k = \frac{\overrightarrow{A_iA_j}}{d(A_i,A_j)}$ and $v_k = \frac{\overrightarrow{B_iB_j}}{d(B_i,B_j)}$ for distinct i, j, k, then explain why there is a unique linear rotation r sending v_2 to u_2. The corresponding affine rotation with center A_0 is called ρ.

   (c) Explain why the definition of angle implies that r also sends v_1 to u_1.

   (d) Find the unique $\lambda > 0$ such that the dilation $h = h_{A_0,\lambda}$ sends $\rho \circ \tau(B_1)$ to A_1.

   (e) We claim that the sought after map is $\phi = h \circ \rho \circ \tau$. Explain why we are done if we can show that $\phi(B_2) = A_2$.

   (f) Prove that if $\phi(B_2)$ lies on both lines A_0A_2 and A_1A_2 then $\phi(B_2) = A_2$.

   (g) Let f be the linear map associated to ϕ. Show that $\angle A_0A_1A_2 = \angle B_0B_1B_2$ implies that $u_1 = f(v_1)$ and likewise prove $u_0 = f(v_0)$.

   (h) Conclude from the previous part that $\overrightarrow{A_0\phi(B_2)}$ and $\overrightarrow{A_0A_2}$ are proportional so that $\phi(B_2)$ does indeed lie on A_0A_2 as claimed.

   (i) Finally prove $\phi(B_2) = A_2$ to conclude the exercise.

4. Pentagon, pentagram, golden section. Recall the golden section is the number $\phi \in \mathbb{R}$ such that $\phi^2 = \phi + 1$ or equivalently $\phi = \frac{1}{\phi - 1}$. It is conceptually attractive as a ratio because: the ratio big/small equals the ratio small/(big-small).

   (a) Choose points A_0 and O in the affine plane and prove that there exists a rotation r with center O such that $r^5 = id$. Is r unique?

   (b) Define $A_i = r^i A_0$ for $i = 1, \ldots 4$. The points A_i form a regular pentagon. Prove that $\phi = d(A_0, A_2)/d(A_0, A_1)$ is the *golden section.*

   (c) What's the composition of isometry and dilation sending our regular pentagon to the smaller pentagon bounded by the segments $[A_i, A_{i+1}]$?