Geometry Tutorial 4

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1. Let E be a (real) Euclidean vector space.

   (a) Prove the Cauchy-Schwarz inequality, i.e.

   $$|x \cdot y| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in E,$$

   where $\|x\| := \sqrt{x \cdot x}$.
   Hint: Consider $\|\lambda x + y\|^2$ for given $x, y \in E$ and $\lambda \in \mathbb{K}$.

   Solution: In the case $x = 0$, this inequality is trivially true. For the case $x \neq 0$ let $\lambda = -\frac{x \cdot y}{\|x\|^2}$. Then we use the following reasoning:

   $$\begin{aligned} 0 &\leq \|\lambda x + y\|^2 \\ &= \lambda^2 (x \cdot x) + (y \cdot y) + 2\lambda x \cdot y \\ &= \frac{(x \cdot y)^2}{\|x\|^4} \|x\|^2 + \|y\|^2 - 2\frac{(x \cdot y)^2}{\|x\|^2} \\ &= \frac{|x \cdot y|^2}{\|x\|^2} + \|y\|^2 - 2\frac{|x \cdot y|^2}{\|x\|^2} \\ &= \|y\|^2 - \frac{|x \cdot y|^2}{\|x\|^2} \end{aligned}$$

   Therefore, $0 \leq \|y\|^2 - \frac{|x \cdot y|^2}{\|x\|^2}$. Rewriting and taking the square root on both sides yields $|x \cdot y| \leq \|x\| \cdot \|y\|$ for all $x, y \in E$.

   (b) Prove that $\| \cdot \|$ defines a norm on E, and that $d(x, y) = \|x - y\|$ makes E into a metric space.

   Solution: It's easy to see that $\|x\| \geq 0$, that $\|x\| = 0 \iff x = 0$ and that $\|\alpha x\| = |\alpha| \|x\|$, by using properties of the inner product. The triangle inequality follows from Cauchy-Schwarz by observing

$$\|x+y\|^2 = (x+y)\cdot(x+y) \le \|x\|^2+2|x\cdot y|+\|y\|^2 \le \|x\|^2+2\|x\|\|y\|+\|y\|^2 = (\|x\|+\|y\|)^2.$$

Here, we have used that $x \cdot y \le |x \cdot y|$. The result follows by taking square roots.

It easily follows from the above that $d(A,B) = \|\overrightarrow{AB}\| \ge 0$, that $d(A,B) = 0 \iff \overrightarrow{AB} = 0 \iff A = B$ and that $d(A,B) = d(B,A)$. Finally, the metric triangle inequality is proven as follows:

$$d(A,C) = \|\overrightarrow{AC}\| = \|\overrightarrow{AB} + \overrightarrow{BC}\| \le \|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = d(A,B) + d(B,C).$$

(c) Prove that $d(A,B) = d(A,C) + d(C,B)$ if and only if A, C and B are collinear, and C is between A and B.

Solution: Looking carefully at our proof of the Cauchy-Schwarz inequality, we see that it becomes an equality if and only if $\|\lambda x + y\|^2 = 0 \iff \lambda x + y = 0$. Recalling the definition of λ gives

$$-\frac{x \cdot y}{\|y\|^2} x + y = 0.$$

So, a necessary condition is that x and y are proportional. To see that this condition is also sufficient, suppose that $y = \mu x$. Then,

$$|x \cdot y| = |\mu| \cdot \|x\| \cdot \|x\| = \|x\| \cdot \|y\|.$$

The triangle inequality for the norm becomes an equality if and only if

$$x \cdot y = \|x\| \cdot \|y\|,$$

i.e. if and only if x and y are proportional *and* in the same direction. To conclude, $d(A,C) = d(A,B) + d(B,C)$ if and only if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are proportional, and in the same direction, which is if and only if A, C and B are collinear, and C is between A and B.

2. Let E be a (finite-dimensional) Euclidean vector space, and let $F \subseteq E$ be a linear subspace. Prove that $E = F \oplus F^\perp$.

Hint: Take an orthonormal basis for F, and use the orthogonal projection P onto F.

Solution: First, note that $x \in F \cap F^\perp$ implies that $x \cdot x = 0$, so $x = 0$.

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis for the subspace F, and let

$$Px = \sum_{i=1}^{k} (x \cdot e_i) e_i$$

be the orthogonal projection. It is clear that $Px \in F$ for all $x \in E$. Moreover,

$$(x - Px) \cdot e_j = x \cdot e_j - \left(\sum_{i=1}^{k} (x \cdot e_i) e_i\right) \cdot e_j = x \cdot e_j - x \cdot e_j = 0,$$

for all $j = 1, 2, \ldots, k$. Hence, $x - Px \in F^{\perp}$. We conclude that $x = Px + (x - Px)$, and the proof is complete.

3. If F and G are complementary subspaces of a vector space E, we define the symmetry s_F about F in the direction of G by $s_F(u + v) = u - v$, for $u \in F$ and $v \in G$.
   If $\mathcal{F}$ is an affine subspace of $\mathcal{E}$, and if $G \subseteq E$ satisfies $E = F \oplus G$, we define $\sigma : \mathcal{E} \to \mathcal{E}$ by $\overrightarrow{O\sigma(M)} = s_F(\overrightarrow{OM})$, where $O \in \mathcal{F}$.
   It can be shown that s_F is a linear involution[1], and that σ is an affine involution, which is independent of the choice of O

   (a) Let H be a hyperplane[2] of a Euclidean vector space E of dimension n, and let x_0 be a nonzero vector of $H^{\perp}$ Prove that

   $$s_H(x) = x - 2\frac{x \cdot x_0}{\|x_0\|^2} x_0.$$

   Solution: Decomposing $x = h + \lambda x_0$, where $h \in H$ and $\lambda \in \mathbb{K}$, we have obviously that

   $$s_H(x) = h - \lambda x_0 = x - 2\lambda x_0,$$

   so it suffices to show that $\lambda = \frac{x \cdot x_0}{\|x_0\|^2}$. Indeed,

   $$x \cdot x_0 = h \cdot x_0 + \lambda x_0 \cdot x_0 = \lambda \|x_0\|^2,$$

   and the result follows.

   (b) Prove that, if x and y are two vectors of the same norm in the Euclidean vector space E, there exists a hyperplane H such that $s_H(x) = y$ (and that H is unique if $x \neq y$).

   Solution: If $\|x\| = \|y\|$, and $x \neq y$, then we can choose $x_0 = x - y \neq 0$. If H is the hyperplane orthogonal to $\text{Span}\{x_0\}$, then by part (a),

   $$\begin{aligned} s_H(x) &= x - 2\frac{x \cdot (x - y)}{(x - y) \cdot (x - y)}(x - y) \\ &= x - 2\frac{\|x\|^2 - x \cdot y}{\|x\|^2 - 2x \cdot y + \|y\|^2}(x - y) \\ &= x - 2\frac{\|x\|^2 - x \cdot y}{2\|x\|^2 - 2x \cdot y}(x - y) = x - (x - y) = y, \end{aligned}$$

[1] An involution $i : A \to A$ is a map which satisfies $i \circ i = \text{id}_A$.
[2] A linear subspace of dimension $n - 1$ (i.e. co-dimension 1).

which proves existence. To see uniqueness, note that a hyperplane is uniquely determined by any nonzero vector in its orthogonal complement. If H' is any hyperplane such that $s_{H'}(x) = y$, and if x_0' is a nontrivial vector in $H^\perp$, we get immediately

$$x_0 = y - x = s_{H'}(x) - x = -2\frac{x \cdot x_0'}{\|x_0'\|^2} x_0' \in \mathrm{Span}\{x_0'\},$$

which shows $H = H'$. Finally, if $x = y$, it's easy to see that $s_H(x) = y$ for any hyperplane H through x.

(c) Prove similarly that, if A and B are two points of an affine space $\mathcal{E}$, there exists an affine hyperplane $\mathcal{H}$ such that $\sigma_{\mathcal{H}}(A) = B$ (and that $\mathcal{H}$ is unique if $A \neq B$).

Solution: If $A \neq B$, then $\overrightarrow{AB}$ is a nonzero vector in E.

4. (a) Let E be a Euclidean vector space. Prove that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in E$. Why is this called the parallelogram identity?

   (b) Prove the polarization identity: $x \cdot y = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right)$ for all $x, y \in E$.

5. Let E be a Euclidean vector space. Prove that any linear isometry is bijective.

Solution: We know every linear isometry is a composition of finitely many reflections. Since every reflection is its own inverse we are done.