Geometry Tutorial 3

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1. Is it true that for any subset S of a Euclidean vector space we have $S = (S^\perp)^\perp$?

> **Solution:**
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> This relation does hold if S is a closed linear subspace of E. In this case the proof goes as follows. ($\subseteq$) If $x \in S$, then x is orthogonal to each $y \in S^\perp$. Therefore, $x \in (S^\perp)^\perp$.
>
> ($\subseteq$) Let $s \in (S^\perp)^\perp$. Note that by a remark in the lecture notes $S \oplus S^\perp = E$. This means we can write $s = u + v$ for $u \in S$ and $v \in S^\perp$. Since $v \in S^\perp$, it is orthogonal to u and s. This means:
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> $$0 = \langle v, s \rangle = \langle v, u + v \rangle = \langle v, u \rangle + \langle v, v \rangle = \langle v, v \rangle$$
>
> This can only be the case when $v = 0$. Therefore, $s = u$ and we have shown that $s \in S$.
>
> In case S is not a closed linear subspace the relation does not hold. Take for instance $S = \{\begin{bmatrix}1\\1\end{bmatrix}\} \subset \mathbb{R}^2$. The orthogonal complement is given by $S^\perp = \text{span}\begin{bmatrix}1\\-1\end{bmatrix}$. Which has as its orthogonal complement $(S^\perp)^\perp = \text{span}\begin{bmatrix}1\\1\end{bmatrix} \neq S$.

2. Imagine a line $\mathcal{L}$ in affine Euclidean plane $\mathcal{E}$ and a vector $u \in \mathcal{L}$ in the direction (underlying vector subspace) of $\mathcal{L}$. The affine map $\gamma_{\mathcal{L},u} : \mathcal{E} \to \mathcal{E}$ defined by $\gamma_{\mathcal{L},u} = t_u \circ \sigma_{\mathcal{L}}$ is called a glide reflection.

   (a) Prove that $\gamma_{\mathcal{L},u}$ is an affine isometry.

   > **Solution:** It is the composition of two affine isometries.

   (b) Why can $\gamma_{\mathcal{L},u}$ not be written as the composition of two reflections?

   > **Solution:** Suppose for the sake of contradiction that $\gamma_{\mathcal{L},u}$ can we written as the composition of:

- 0 reflections: Then $\gamma_{\mathcal{L},u}$ is the identity.
- 1 reflection: Then, $\gamma_{\mathcal{L},u}$ is an involution.
- 2 reflections: There are two options here. Either the two mirror lines intersect, in which case $\gamma_{\mathcal{L},u}$ has a fixed point, or the two mirror lines are parallel, in which case $\gamma_{\mathcal{L},u}$ is a pure translation.

Each of these decompositions leads to an obvious contradiction.

(c) Write $\gamma_{\mathcal{L},u}$ explicitly as a composition of three reflections.

Solution: This boils down to decomposing the translation t_u as a composition of two reflections. To see this, pick a point $O \in \mathcal{E}$, and let $F = \{u\}^\perp$ and $O' \in \mathcal{E}$ such that $\overrightarrow{OO'} = \frac{u}{2}$. If $\mathcal{F}$ is the line through O with direction F and $\mathcal{F}'$ is the line through O' parallel to $\mathcal{F}$, we claim that $t_u = \sigma_{\mathcal{F}} \circ \sigma_{\mathcal{F}'}$. This follows from the simple calculation

$$\begin{aligned}
\overrightarrow{M\sigma_{\mathcal{F}} \circ \sigma_{\mathcal{F}'}(M)} &= \overrightarrow{MO} + \overrightarrow{O\sigma_{\mathcal{F}} \circ \sigma_{\mathcal{F}'}(M)} \\
&= \overrightarrow{MO} + s_F\left(\overrightarrow{O\sigma_{\mathcal{F}'}(M)}\right) \\
&= \overrightarrow{MO} + s_F\left(\overrightarrow{OO'} + \overrightarrow{O'\sigma_{\mathcal{F}'}(M)}\right) \\
&= \overrightarrow{MO} + s_F\left(\overrightarrow{OO'}\right) + s_F \circ s_F\left(\overrightarrow{O'M}\right) \\
&= \overrightarrow{MO} - \overrightarrow{OO'} + \overrightarrow{O'M} = u.
\end{aligned}$$

3. Verify that $s_F \circ s_F$ is the identity. Give an example of linear subspaces $F, G \subset E$ such that $s_F \circ s_G \neq s_G \circ s_F$.

Solution: For the first question, let $x \in F$ and $y \in F^\perp$ then $s_F \circ s_F(x+y) = s_F(x-y) = x+y$. This shows that s_F must be the identity map. Taking $F = \operatorname{Span} e_1$ and $G = \operatorname{Span} e_1 + e_2$ we find $s_F \circ s_G \neq s_G \circ s_F$. For example $s_F \circ s_G(e_1) = -e_2$ and $s_G \circ s_F(e_1) = e_2$.