Geometry: Tutorial II

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1. Is Pappus' theorem valid if the lines $\mathscr{D}, \mathscr{D}'$ are on two distinct lines in three-dimensional affine space?

> **Solution:** If $\mathscr{D}'$ and $\mathscr{D}$ are in distinct planes, the two sets of parallel lines cannot exist as in the theorem. So Pappus' theorem is true in any dimension ≥ 2.

2. (a) Prove that the composition of two affine maps ϕ and ψ is again an affine map, and that the linear map associated to $\psi \circ \phi$ is the composition of the linear maps associated to ψ and ϕ, i.e. that $\overrightarrow{\psi \circ \phi} = \overrightarrow{\psi}\,\overrightarrow{\phi}$.

> **Solution:** Upon picking an origin O, we get
>
> $$\overrightarrow{O\psi(\phi(M))} = \overrightarrow{\psi}\left(\overrightarrow{O\phi(M)}\right) = \overrightarrow{\psi}\,\overrightarrow{\phi}\left(\overrightarrow{OM}\right),$$
>
> which immediately gives the desired result.

(b) Let $\phi \in GA(\mathscr{E})$,[1] and let $h(O, \lambda)$ be a central dilation. Compute $\phi \circ h(O, \lambda) \circ \phi^{-1}$.

> **Solution:** If f is the linear map associated to ϕ, it follows from the previous exercise that the composition is an affine map, with associated linear map $f(\lambda I)f^{-1} = \lambda I$. Hence,
>
> $$\overrightarrow{O\phi \circ h(O, \lambda) \circ \phi^{-1}(M)} = \lambda I\overrightarrow{OM} = \lambda\overrightarrow{OM},$$
>
> and thus $\phi \circ h(O, \lambda) \circ \phi^{-1} = h(O, \lambda)$.

(c) Prove that two dilations with the same center commute.

[1] An invertible affine map from $\mathscr{E}$ to itself.

Solution: This is most easily shown by proving that the composition of $h(O, \lambda)$ and $h(O, \lambda')$ is equal to $h(O, \lambda\lambda')$. The result then follows from interchanging λ and λ', and from commutativity of multiplication in $\mathbb{K}$. Indeed,

$$\overrightarrow{Oh(O,\lambda) \circ h(O,\lambda')(M)} = \lambda \overrightarrow{Oh(O,\lambda')(M)} = \lambda' \lambda \overrightarrow{OM},$$

and we are done.

(d) Compute $h(B, \lambda') \circ h(A, \lambda)$.

Solution: Let $C \in \mathscr{E}$. We have

$$\begin{aligned}
\overrightarrow{Ch(B,\lambda') \circ h(A,\lambda)(M)} &= \overrightarrow{CB} + \overrightarrow{Bh(B,\lambda') \circ h(A,\lambda)(M)} \\
&= \overrightarrow{CB} + \lambda' \overrightarrow{Bh(A,\lambda)(M)} \\
&= \overrightarrow{CB} + \lambda' \overrightarrow{BA} + \lambda' \overrightarrow{Ah(A,\lambda)(M)} \\
&= \overrightarrow{CB} + \lambda' \overrightarrow{BA} + \lambda\lambda' \overrightarrow{AM}.
\end{aligned}$$

Let's try to find out if the composition has a fixed point. Choosing $C = B$, a point M is a fixed point if

$$\overrightarrow{Bh(B,\lambda') \circ h(A,\lambda)(M)} = \lambda' \overrightarrow{BA} + \lambda\lambda' \overrightarrow{AM} = \overrightarrow{BM}.$$

Since $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$, we can solve the above equation for $\overrightarrow{BM}$:

$$\lambda' \overrightarrow{BA} + \lambda\lambda' \overrightarrow{AB} + \lambda\lambda' \overrightarrow{BM} = \overrightarrow{BM} \iff \overrightarrow{BM} = \frac{\lambda'(\lambda - 1)}{1 - \lambda\lambda'} \overrightarrow{AB},$$

provided $\lambda\lambda' \neq 1$. In this case, we let C be the *unique* point such that $\overrightarrow{BC} = \frac{\lambda'(\lambda-1)}{1-\lambda\lambda'} \overrightarrow{AB}$. We claim that the composition is a dilation with center C and ratio $\lambda\lambda'$.

Indeed,

$$\begin{aligned}
\overrightarrow{Ch(B,\lambda') \circ h(A,\lambda)(M)} &= \overrightarrow{CB} + \lambda' \overrightarrow{BA} + \lambda\lambda' \overrightarrow{AM} \\
&= \frac{\lambda'(1-\lambda)}{1-\lambda\lambda'} \overrightarrow{AB} + \lambda' \overrightarrow{BA} + \lambda\lambda' \overrightarrow{AM} \\
&= \left(\frac{\lambda'(1-\lambda)}{1-\lambda\lambda'} \overrightarrow{AB} - \lambda' \overrightarrow{AB} + \lambda\lambda' \overrightarrow{AC} \right) + \lambda\lambda' \overrightarrow{CM}
\end{aligned}$$

It can be shown that the term in brackets equals zero, by writing $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, and by using the defining equation. This shows that the composition is a dilation with center C and ratio $\lambda\lambda'$. It is worth noting that the center of the new dilation is on the line through A and B.

Finally, we have to consider the case $\lambda\lambda' = 1$. In this case, the associated linear mapping is the identity, so we expect the composition to be a translation. Indeed, choosing $C = M$, we see that

$$\overrightarrow{Mh(B,\lambda')\circ h(A,\lambda)(M)} = \overrightarrow{MB} + \lambda'\overrightarrow{BA} + \overrightarrow{AM} = (1-\lambda')\overrightarrow{AB},$$

independent of M. Thus, the composition is a translation over the vector $(1-\lambda')\overrightarrow{AB}$. Again, it's worth noting that the composition translates in the direction of the line through A and B.

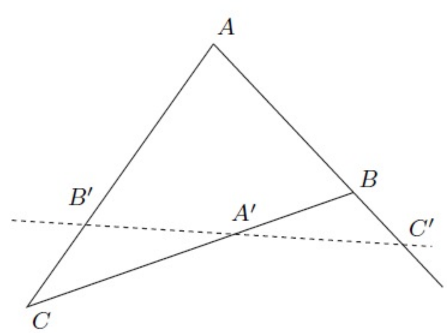
(e) Explain whether or not the set of all dilations is a subgroup of $GA(\mathscr{E})$.

Solution: No, if $\lambda\lambda' = 1$, then the composition of two dilations is a translation. The set of all translations and dilations *does* form a subgroup of $GA(\mathscr{E})$ (even a normal subgroup!).

3. In this exercise we aim to prove **Menelaüs' theorem** in a couple of steps, see also Audin Exercise I.37. In an affine plane, consider three distinct non-collinear points A, B, C and points A' on line BC, B' on CA and C' on AB all distinct from A, B, C. Points A', B', C' are collinear if and only if $\frac{\overrightarrow{A'B}}{\overrightarrow{A'C}} \cdot \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}} \cdot \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}} = 1$.

(a) Draw a picture.

Solution:



(b) Explain why $h\left(C', \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}\right)$ sends B to A and $h\left(B', \frac{\overrightarrow{B'C}}{\overrightarrow{B'A}}\right)$ sends A to C and $h\left(A', \frac{\overrightarrow{A'B}}{\overrightarrow{A'C}}\right)$ sends C to B.

Solution:
The dilation map is the map $h(O,\lambda)$ such that $\overrightarrow{Oh(O,\lambda)(M)} = \lambda\overrightarrow{OM}$. This means that:

$$\overrightarrow{C'h\left(C', \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}\right)(B)} = \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}\overrightarrow{C'B} = \overrightarrow{C'A}.$$

Hence, we see that $h\left(C', \frac{\overrightarrow{C'A}}{\overrightarrow{C'B}}\right)(B) = A$. Similarly, for the other cases.

(c) The composition of the three maps above sends B to itself. Why must the composition be of the form $h(B, \lambda)$ for some $\lambda \in \mathbb{R}$?

Solution: By part (d) of the previous question, we know that a composition of dilations is either a translation or a dilation. If the composition has a fixed point B, as is the case here, must hence be a dilation of the form $h(B, \lambda)$.

(d) Show that the first two maps fix[2] the line $C'B'$.

Solution: Let M be a point on the line $C'B'$ then $\overrightarrow{B'M}$ is proportional to $\overrightarrow{C'B'}$ i.e $\overrightarrow{B'M} = \mu\overrightarrow{C'B'}$. If we apply the dilation $h(B', \lambda)$ to M we get $\overrightarrow{B'h(B', \lambda)(M)} = \lambda\overrightarrow{B'M} = \lambda\mu\overrightarrow{C'B'}$. This shows that $h(B', \lambda)(M)$ is on the line $C'B'$.

(e) Show that the third map fixes $C'B'$ if and only if A' is on that line.

Solution: ($\Rightarrow$) Suppose that $h(A', \lambda)$ fixes $B'C'$. This means that $h(A', \lambda)(B')$ is on $B'C'$, therefore we have that $\overrightarrow{A'h(A', \lambda)(B')}$ is proportional to $\overrightarrow{B'C'}$ i.e $\overrightarrow{A'h(A', \lambda)(B')} = \mu\overrightarrow{B'C'}$. Computing the left hand side, we obtain $\lambda\overrightarrow{A'B'} = \mu\overrightarrow{B'C'}$. This shows that A' is on the line $B'C'$, where we have to use that $\lambda \neq 0$ (in this case $h(A', 0)$ would be a constant map, which could never fix the line $B'C'$).

($\Leftarrow$) This follows immediately from the calculation in part (d).

(f) Conclude that if the composition of the three is the identity, then A', B', C' are collinear.

Solution: The identity transformation fixes $B'C'$, so A' lies on $B'C'$ by part (e) and hence A', B' and C' are collinear.

(g) Show that the point B does not lie on line $A'C'$, because otherwise, $\overrightarrow{AB}$ would be proportional to $\overrightarrow{BC}$, making the sides of the triangle linearly dependent.

Solution: Suppose that B does lie on the line $A'C'$. Then $\overrightarrow{A'B}$ and $\overrightarrow{C'B}$ would be proportional. Since A' is a point on BC we have that $\overrightarrow{A'B}$ is proportional to $\overrightarrow{BC}$. Similarly, since C' is on AB, we have that $\overrightarrow{C'B}$ is proportional to $\overrightarrow{AB}$. Therefore,

[2] This means that a point which is on line $C'B'$ stays on the line $C'B'$ after applying the map.

if $A'B$ and $C'B$ would be proportional, this would mean AB and BC would also be proportional, which cannot be the case in a triangle. Therefore, we have found a contradiction and we have shown that B does not lie on $A'C'$.

(h) Conversely, if A', B', C' are collinear, then show the composition of the three is the identity.

Solution: Suppose A', B' and C' are collinear. Then by part (e) the composition of the three maps fixes $B'C'$. The composition of the three maps is of the form $h(B, \lambda)$ by part (c). Moreover, we know that the point B does not lie on $B'C'$. Therefore, the composition given by $h(B, \lambda)$ must be the identity map.