Geometry Tutorial 10

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1. Let $X, Y : \mathbb{R}^2 \to \mathbb{R}$ with $X(x, y) = (0, x)$, $Y(x, y) = (y, 0)$ and $H(x, y) = (x, -y)$. Show that for the commutator $[X, Y]\partial_X Y - \partial_Y X$ we have that $[X, Y] = H$, $[X, H] = -2X$ and $[Y, H] = 2Y$.

2. Prove that if γ is a geodesic in a Riemannian chart then $|\dot{\gamma}(t)|$ must be constant. Also give an example to illustrate that the converse is not true: there are non-geodesic constant speed curves.

3. Verify that the Δ from the proof of Lemma 6.5 in the lecture notes is indeed an LC- connection in P by checking it satisfies all the required axioms.

4. Check that the Riemann curvature satisfies $R(X, Y)Z = -R(Y, X)Z$ for all vector fields X, Y, Z.

5. (Parallel transport) We say a vector field X on chart (P, g) is parallel along curve $\gamma : [0, 1] \to P$ if $\nabla_{\dot{\gamma}} X = 0$.
   (a) Taking γ as before, show that for any $p \in P$ and any $v \in \mathbb{R}^n$ there exists a parallel vector field X. In what sense is it unique?
   (b) Prove that a curve γ is a geodesic if and only if $\dot{\gamma}$ is a parallel along γ.
   (c) We say w is the parallel translate along v if $X(0) = v$ and $X(1) = w$ for some vector field parallel along γ.
   (d) Stretch your arms vertically upwards and make fists with your thumbs pointing towards each other. Now rotate your right arm to your right side so it sticks out horizontally while keeping it stretched out. Next rotate your right arm to point directly in front of you while keeping wrist and thumb in position. Finally rotate your left arm downwards to again be parallel to your right arm and compare the positions of your thumbs.
   (e) Explain what the previous physical exercise has to do with parallel translation on a chart of the sphere.

6. Compute the Riemannian curvature of the hyperbolic plane $\mathbb{H}^2$ and verify that the scalar curvature is constantly -2.