Geometry Tutorial 13

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1. Let $X, Y : \mathbb{R}^2 \to \mathbb{R}$ with $X(x,y) = (0,x)$, $Y(x,y) = (y,0)$ and $H(x,y) = (x,-y)$. Show that for the commutator $[X,Y] \coloneqq \frac{\partial X}{\partial Y} - \frac{\partial Y}{\partial X}$ we have that $[X,Y] = H$, $[X,H] = -2X$ and $[Y,H] = 2Y$.

Solution:

$$\begin{aligned}
[X,Y] &= \partial_X Y - \partial_Y X \\
&= Y'(p)X(p) - X'(p)Y(p) \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x \\ -y \end{pmatrix} \\
&= H(x,y).
\end{aligned}$$

$$\begin{aligned}
[X,H] &= \partial_X H - \partial_H X \\
&= H'(p)X(p) - X'(p)H(p) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -2x \end{pmatrix} \\
&= -2X.
\end{aligned}$$

$$\begin{aligned}[Y,H] &= \partial_Y H - \partial_H Y \\ &= H'(p)Y(p) - Y'(p)H(p) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \\ &= \begin{pmatrix} 2y \\ 0 \end{pmatrix} \\ &= 2Y. \end{aligned}$$

2. Prove that if γ is a geodesic in a Riemannian chart then $|\dot{\gamma}(t)|$ must be constant. Also give an example to illustrate that the converse is not true: there are non-geodesic constant speed curves.

Solution: Notice that $|\dot{\gamma}(t)| = \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}$. For simplicity we remove the square root by squaring the expression. We note that $|\dot{\gamma}(t)|$ is constant if and only if the first time derivative is zero. Hence, we compute the first derivative w.r.t. t:

$$\begin{aligned}\frac{d}{dt}|\dot{\gamma}(t)|^2 &= \frac{d}{dt} g(\dot{\gamma}(t), \dot{\gamma}(t) \\ &= g(\gamma(t))(\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t), \dot{\gamma}(t)) + g(\gamma(t))(\dot{\gamma}(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)) \\ &= 2g(\gamma(t))(\nabla_{\dot{\gamma}(t)}(\dot{\gamma}(t)), \dot{\gamma}(t)).\end{aligned}$$

Where the third step is by the third property of the Levi-Civita connection. Since γ is a geodesic, we know that $\nabla_{\dot{\gamma}(t)}(\dot{\gamma}(t)) = 0$. Proving that the curve must have constant speed.

An example of a non-geodesic curve with constant speed is a circle in $\mathbb{R}^2$. This curve would have constant speed, but is not a geodesic, because under the standard metric it is not the shortest path between two points.

3. Verify that the Δ from the proof of Lemma 6.5 in the lecture notes is indeed an LC- connection in P by checking it satisfies all the required axioms.

Solution:

4. Check that the Riemann curvature satisfies $R(X,Y)Z = -R(Y,X)Z$ for all vector fields X, Y, Z.

Solution: Starting from the right-hand side:

$$\begin{aligned} R(Y,X)Z &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \\ &= -(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) - \nabla_{-[X,Y]} Z \\ &= -(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = -R(X,Y)Z, \end{aligned}$$

where I have used that $\nabla_{fX} Z = f \nabla_X Z$.

5. (Parallel transport) We say a vector field X on chart (P, g) is parallel along curve $\gamma : [0,1] \to P$ if $\nabla_{\dot\gamma} X = 0$.

(a) Taking γ as before, show that for any $p \in P$ and any $v \in \mathbb{R}^n$ there exists a parallel vector field X. In what sense is it unique?

Solution: Assuming that $\gamma(0) = p$, and $v \in \mathbb{R}^n$ is a tangent vector at p, the parallel transport equation $\nabla_{\dot\gamma} X = 0$ defines a system of n ODEs for the coefficients of X. The vector v specifies the initial conditions. So, by the existence and uniqueness theorem for ODEs, there exists a unique parallel vector field X, which is defined (at least) on the restriction $\gamma([0,\epsilon)) \subseteq P$.

(b) Prove that a curve γ is a geodesic if and only if $\dot\gamma$ is a parallel along γ.

Solution: This is immediate from the fact that the geodesic equation can be written as $\nabla_{\dot\gamma}\dot\gamma = 0$.

(c) We say w is the parallel translate along v if $X(0) = v$ and $X(1) = w$ for some vector field parallel along γ. Show that $g(\gamma(0))(v, \dot\gamma(0)) = g(\gamma(1))(w, \dot\gamma(1))$, whenever γ is a geodesic.

Solution: Consider the function

$$\alpha : [0,1] \to \mathbb{R} : t \mapsto g(\gamma(t))(X(t), \dot\gamma(t)).$$

It suffices to prove that this function is constant along solutions of the parallel transport equation. Using the third property of the LC-connection on (P, g), we can write

$$\frac{d}{dt}\alpha(t) = \partial_{\dot\gamma(t)} g(\gamma(t))(X(t), \dot\gamma(t)) = g(\gamma(t))(\nabla_{\dot\gamma(t)} X(t), \dot\gamma(t)) + g(\gamma(t))(X(t), \nabla_{\dot\gamma(t)}\dot\gamma(t))$$

The first term is zero since X is parallel along γ, while the second term is zero (**provided γ is a geodesic**). Hence, we are done.

(d) Stretch your arms vertically upwards and make fists with your thumbs pointing towards each other. Now rotate your right arm to your right side so it sticks out horizontally while keeping it stretched out. Next rotate your right arm to point directly in front of you while keeping wrist and thumb in position. Finally rotate your left arm downwards to again be parallel to your right arm and compare the positions of your thumbs.

Solution: Follow the instructions.

(e) Explain what the previous physical exercise has to do with parallel translation on a chart of the sphere.

Solution: It shows that parallel transport depends on the path taken. We start in a situation where the thumbs (i.e. tangent vectors) are opposite, and we end in a situation where they are orthogonal. Also, parallel transport along a closed loop does not have to return the same vector.

Remark: This is another manifestation of curvature. By writing down the parallel transport equation in Euclidean space, you can convince yourself that parallel transport along any closed loop is the identity operation.

6. Compute the Riemannian curvature of the hyperbolic plane $\mathbb{H}^2$ and verify that the scalar curvature is constantly -2.

Solution:

Unfortunately, this is a pretty long calculation. It's not hard, but it's very easy to miss a minus sign, and mess up the answer. I will give the intermediate answers, and some tips to speed up the calculation.

For the metric, we have

$$g_{ij} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}, \quad g_{ij}^{-1} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix},$$

simply be definition. Notice that the metric tensor is diagonal, so we can immediately ignore any terms involving g_{12} or g_{12}^{-1}. Recall the equation for the Christoffel symbols:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_s g_{ks}^{-1} \left(\partial_i g_{js} + \partial_j g_{is} - \partial_s g_{ij} \right).$$

Notice that, given a value for k, there is always exactly one s which contributes, namely $s = k$. We can also ignore any terms involving ∂_1, and use symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$. This gives

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = -\frac{1}{y}, \quad \Gamma_{11}^2 = +\frac{1}{y},$$

with all other combinations being equal to zero.

Next, recall the definition of the Riemann curvature tensor:

$$R_{ij,k}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum_s \Gamma_{jk}^s \Gamma_{is}^l - \sum_s \Gamma_{ik}^s \Gamma_{js}^l.$$

This time, there is a skew-symmetry $R^l_{ij,k} = -R^l_{ji,k}$. Also, given j and k, there is always exactly one s which contributes. To see this, note that $\Gamma^s_{jk} \neq 0$ for precisely one s. Hence, we never actually have to compute a sum. For the terms involving $\partial\Gamma$, it's easy to see at a glance when these are zero. Almost all of them involve either a ∂_1 or a vanishing Christoffel symbol. Keeping the above in mind, we find that

$$R^2_{12,1} = R^1_{21,2} = +\frac{1}{y^2}, \quad R^1_{12,2} = R^2_{21,1} = -\frac{1}{y^2}.$$

The final step is to compute the traces. Luckily, this is pretty simple. By definition, we have:

$$R_{ij} = \sum_k R^k_{ki,j},$$

and hence

$$R_{11} = R^1_{11,1} + R^2_{21,1} = -\frac{1}{y^2}, \quad R_{22} = -\frac{1}{y^2},$$

while the diagonal terms are zero. The scalar curvature is thus

$$S = \sum_{i,j} g^{-1}_{ij} R_{ij} = g^{-1}_{11} R_{11} + g^{-1}_{22} R_{22} = -2,$$

as required!

Remark: The Riemann curvature tensor has some extra hidden symmetries. These become more apparent when we lower the final index using the metric tensor:

$$R_{ij,kl} = \sum_n g^{-1}_{nl} R^n_{ij,k}.$$

It turns out that for two dimensional charts, the above tensor only has one independent component, namely $R_{12,12}$. We can rewrite the scalar curvature as

$$\frac{2R_{12,12}}{\det(g)},$$

where g is the matrix representing the metric tensor.