Geometry Tutorial 12

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7-2 2020

1. Describe qualitatively the geodesics on:
    1. A spheroid (the surface obtained by rotating an ellipse around one of its axes).
    2. A cylinder.
    3. A torus.
    4. A cone
2. The shortest path between two points may not always take you along a geodesic. For example prove that in the Riemannian chart (P, g_E) with $P = \mathbb{R}^2 - \{0\}$ and $g_E(p)$ the standard Euclidean inner product at all points there does NOT exist a geodesic connecting $-e_1$ to e_1.
3. Minimize the length of a curve in the Euclidean plane directly, do you still find straight lines as solutions?
4. Show that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Why is this useful for calculating Christoffel symbols?
5. (a) Compute the Christoffel symbols of the sphere in spherical coordinates.
    (b) Show that the great circles that are meridians are geodesics on the sphere.
    (c) Explain why moving from point $p = (1, 0, 0)$ to $q = (0, 1, 0)$ along a great circle may not be the shortest path from p to q on the sphere.
    (d) Why does this actually prove that all great circles on the sphere are geodesics?
6. (a) For vector fields $X, Y, Z : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $X(x, y, z) = (x, 0, 0)$, $Y(x, y, z) = (x, y, z)$ and $Z(x, y, z) = e_2 + xe_3$ compute $H = \partial_X Y$ and $\partial_Z H$. Also compute $[X, [Y, Z]]$.
    (b) Verify the Jacobi identity in the above example
    $$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$
    (c) Prove the Jacobi identity for any vector fields X, Y, Z on $\mathbb{R}^n$.