Geometry Tutorial 12

Martijn Kluitenberg and Oscar Koster

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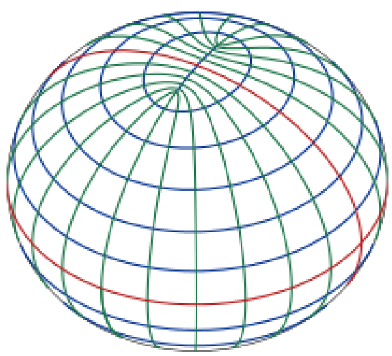
1. Describe qualitatively the geodesics on:

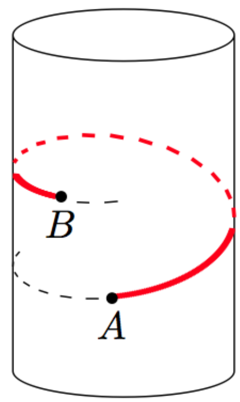
   1. A spheroid (the surface obtained by rotating an ellipse around one of its axes).

   Solution:
   Geodesics on an ellipsoid can be depicted as follows:

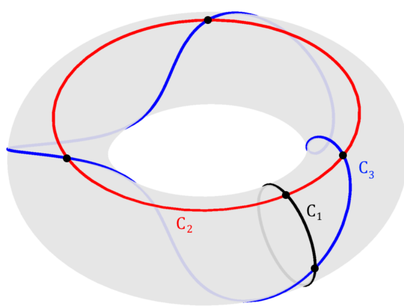
   2. A cylinder.

   Solution: Geodesics on a Cylinder are the following curves:



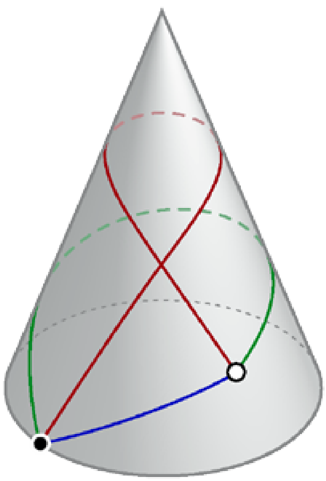
3. A torus.

Solution: There are three types of geodesics on a torus:



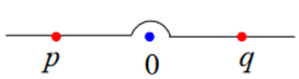
4. A cone.

Solution: The geodesics on a cone are:



2. The shortest path between two points may not always take you along a geodesic. For example prove that in the Riemannian chart (P, g_E) with $P = \mathbb{R}^2 - \{0\}$ and $g_E(p)$ the standard Euclidean inner product at all points there does NOT exist a geodesic connecting $-e_1$ to e_1.

Solution: Let $p = (-1, 0)$ and $q = (1, 0)$. The shortest path between these two points would be the straight line connecting p and q. However, it is clear that this path crosses the origin, which is not part of the space. So any other short path between p and q would be a straight line going closer and closer to the origin, before making a small 'jump' over the origin and continuing in a straight line. This can always be improved by making the 'jump' closer to the origin, meaning there always is a shorter path. This means there is no geodesic between p and q.



3. Minimize the length of a curve in the Euclidean plane directly, do you still find straight lines as solutions?

Solution: Suppose that we have a curve $\gamma : [a, b] \to \mathbb{R}^2$ connecting two points (a, c) en (b, c) in $\mathbb{R}^2$. Note that we can assume w.l.o.g. that the two points are on the same horizontal line.

Let us parametrize the curve by arc-length, which means that we can write $L(\gamma) = \int_a^b \sqrt{1+\gamma'(t)^2}dt$. Notice that $\sqrt{1+\gamma'(t)^2} \geq 0$ and continuous, hence minimizing the integral means we need to minimize the integrand. Taking the derivative and setting it equal to zero tells us that the integrand is minimized only if $\gamma'(t) = 0$ i.e. if $\gamma(t)$ is a constant curve, which is the straight horizontal line connecting (a, c) en (b, c).

4. Show that $\Gamma_{ij}^k = \Gamma_{ji}^k$. Why is this useful for calculating Christoffel symbols?

Solution: Notice that $g_{ij}(p) = g_{ji}(p)$ by symmetry of the inner product. Using this and changing the order of summation, we can write:

$$\begin{aligned}\Gamma_{ij}^k &= \frac{1}{2}\sum_k g_{rk}^{-1}(p)\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right)\\ &= \frac{1}{2}\sum_k g_{rk}^{-1}(p)\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ji}}{\partial x_k}\right)\\ &= \Gamma_{ji}^k.\end{aligned}$$

This is useful in computing the Christoffel symbols because we do not have to compute Γ_{ij}^k and Γ_{ji}^k separately, reducing the number of symbols to compute by a half.

5. (a) Compute the Christoffel symbols of the sphere in spherical coordinates.

Solution: We can parametrize a sphere in spherical coordinates by the map:

$$\gamma(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

First, we need to compute the Jacobian of the map, which yields:

$$\gamma'(\rho, \theta, \phi) = \begin{pmatrix} \cos\theta\sin\phi & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\cos\theta\sin\phi \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}.$$

Next,we compute the pull-back metrics.

$$
\begin{aligned}
g_{11}(p) &= \gamma^* g_{Eucl.}(p)(e_1, e_1)\\
&= g_E\left(\begin{pmatrix}\cos\theta\sin\phi\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}, \begin{pmatrix}\cos\theta\sin\phi\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}\right)\\
&= \cos^2\theta\sin^2\phi + \sin^2\phi\sin^2\theta + \cos^2\phi\\
&= \sin^2\phi + \cos^2\phi\\
&= 1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
g_{12}(p) &= \gamma^* g_{Eucl.}(p)(e_1, e_2)\\
&= g_E\left(\begin{pmatrix}\cos\theta\sin\phi\\ \sin\phi\sin\theta\\ \cos\phi\end{pmatrix}, \begin{pmatrix}\rho\cos\phi\cos\theta\\ \rho\sin\theta\cos\phi\\ -\rho\sin\phi\end{pmatrix}\right)\\
&= \rho\sin\phi\cos\phi\cos^2\theta + \rho\sin\phi\cos\phi\sin^2\theta - \rho\sin\phi\cos\phi\\
&= \rho\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta) - \rho\sin\phi\cos\phi\\
&= 0.
\end{aligned}
$$

In a similar way (or by mathematica) the rest can be computed. The results can be summarised in the following matrix:

$$
g_{ij} = \begin{pmatrix}1 & 0 & 0\\ 0 & \rho^2 & 0\\ 0 & 0 & \rho^2\sin^2\phi\end{pmatrix}.
$$

This has the inverse:

$$
g_{ij}^{-1} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1/\rho^2 & 0\\ 0 & 0 & 1/\rho^2\sin^2\phi\end{pmatrix}.
$$

Using this, we can compute the Christoffel symbols:

$$
\begin{aligned}
\Gamma^1_{11} =& \frac{1}{2}\sum_{k=1}^{3} g_{1k}^{-1}(p)\left(\frac{\partial g_{1k}}{\partial x_1} + \frac{\partial g_{1k}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_k}\right)\\
=& \frac{1}{2}\left(g_{11}^{-1}(p)\left(2\frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}\right)\right.\\
&+ g_{12}^{-1}(p)\left(2\frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right)\\
&\left.+ g_{13}^{-1}(p)\left(2\frac{\partial g_{13}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_3}\right)\right.\\
=& 0.
\end{aligned}
$$

Where we note that g_{1k} is either constant or zero, and hence its partial derivatives will be zero with respect to any variable.
Similarly we can compute for example,

$$\begin{aligned}
\Gamma_{12}^2 =& \frac{1}{2}\sum_{k=1}^{3} g_{2k}^{-1}(p)\left(\frac{\partial g_{2k}}{\partial x_1} + \frac{\partial g_{1k}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_k}\right)\\
=& \frac{1}{2}\left(g_{21}^{-1}(p)\left(\frac{\partial g_2 1}{\partial x_1} + \frac{\partial g_{11}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_1}\right)\right.\\
&+ g_{22}^{-1}(p)\left(\frac{\partial g_2 2}{\partial x_1} + \frac{\partial g_{1k}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_2}\right)\\
&\left.+ g_{23}^{-1}(p)\left(\frac{\partial g_2 3}{\partial x_1} + \frac{\partial g_{13}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_3}\right)\right)\\
=& \frac{1}{2} g_{22}^{-1}(p)\left(\frac{\partial g_{22}}{\partial x_1}\right)\\
=& \frac{1}{2}\frac{1}{\rho^2}\frac{\partial \rho^2}{\partial \rho}\\
=& \frac{2\rho}{\rho^2}\\
=& \frac{1}{\rho}.
\end{aligned}$$

The rest of the Christoffel symbols can be computed in a similar way or by using the Mathematica file that is on the website. The resulting Christoffel symbols can be summarised in the following matrices:

$$\Gamma_{ij}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & -\rho\sin^2\phi \end{pmatrix}$$

$$\Gamma_{ij}^2 = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & -\cos\phi\sin\phi \end{pmatrix}$$

$$\Gamma_{ij}^3 = \begin{pmatrix} 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & \frac{\cos\phi}{\sin\phi} \\ \frac{1}{\rho} & \frac{\cos\phi}{\sin\phi} & 0 \end{pmatrix}$$

(b) Show that the great circles that are meridians are geodesics on the sphere.

Solution: Consider the spherical coordinates $\gamma = (r, \theta, \phi)$. We need to rewrite the geodesic equations to the following (noticing the rest of the Christoffel symbols are zero):

$$\begin{aligned}\ddot{\gamma_1} &= -\Gamma^1_{22}(\dot{\gamma_2})^2 - \Gamma^1_{33}(\dot{\gamma_3})^2 \\ &= \rho\dot{\theta}^2 + \rho\sin^2\theta\dot{\phi}^2\end{aligned}$$

Similarly,

$$\begin{aligned}\ddot{\gamma_2} &= -2\Gamma^2_{21}(\dot{\gamma_2}\dot{\gamma_1}) - \Gamma^2_{33}(\dot{\gamma_3})^2 \\ &= \frac{-2}{\rho}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}\end{aligned}$$

and,

$$\begin{aligned}\ddot{\gamma_3} &= -2\Gamma^3_{13}(\dot{\gamma_1}\dot{\gamma_3}) - 2\Gamma^3_{23}(\dot{\gamma_2}\dot{\gamma_3}) \\ &= \frac{-2}{\rho}\dot{r}\dot{\phi} - 2\cot\theta\dot{\theta}\dot{\phi}.\end{aligned}$$

Let us consider a unit sphere for simplicity. A meridian is a curve on the sphere that keeps one of the angles constant and changes the other one by some constant speed. This means we can write a geodesic as $\phi = \phi_0$ and $\theta = kt + \theta_0$. In this case, note that $\ddot{\phi} = \ddot{\theta} = 0$, $\dot{\theta} = \theta_0$ and $\dot{\phi} = 0$. Plugging this in the the geodesic equations, we obtain that indeed for these curves $\ddot{\phi}$ and $\ddot{\theta}$ are equal to zero, and hence meridians are geodesics of the sphere.

(c) Explain why moving from point $p = (1, 0, 0)$ to $q = (0, 1, 0)$ along a great circle may not be the shortest path from p to q on the sphere.

Solution: Note that on a sphere there are two directions in which you can walk along a great circle from $p = (1, 0, 0)$ to $q = (0, 1, 0)$. One way is the the shortest path, the other path is first going to $q' = (0, -1, 0)$ and then to $q = (0, 1, 0)$, which has length greater then or equal to the first path.

6. (a) For vector fields $X, Y, Z : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $X(x, y, z) = (x, 0, 0)$, $Y(x, y, z) = (x, y, z)$ and $Z(x, y, z) = e_2 + xe_3$ compute $H = \partial_X Y$ and $\partial_Z H$. Also compute $[X, [Y, Z]]$.

Solution:

$$\begin{aligned} H = \partial_X Y & \\ &= Y'(p)X(p) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \partial_Z H &= H'(p)Z(p) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} [X,[Y,Z]] &= \partial_X[Y,Z] - \partial_{[Y,Z]}X \\ &= \partial_X(\partial_Y Z - \partial_Z Y) - X'(p)(\partial_Y Z - \partial_Z Y) \\ &= \partial_X\left(\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

(b) Verify the Jacobi identity in the above example

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Solution: Notice that we can compute $[Z,X] = 0$ and $[X,Y] = 0$. Therefore, $[Y,[Z,X]] = [Z,[X,Y]] = 0$, and hence $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0$. Proving the Jacobi identity is satisfied.

(c) Prove the Jacobi identity for any vector fields X, Y, Z on $\mathbb{R}^n$.

Solution:
Let X, Y and Z be any vector field on $\mathbb{R}^n$.

$$\begin{aligned} [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] &= \partial_X[Y,Z] - \partial_[Y,Z]X + \partial_Y[Z,X] - \partial_[Z,X]Y + \partial_Z[X,Y] - \partial_[X,Y]Z \\ &= \partial_X\partial_Y Z - \partial_Y\partial_X Z - \partial_{[X,Y]}Z + \partial_Z\partial_X Y - \partial_X\partial_Z Y - \partial_{[Z,X]}Y + \partial_Y\partial_Z X \end{aligned}$$

Note that lemma 6.4 in the lecture notes states that $\partial_X \partial_Y Z - \partial_Y \partial_X Z - \partial_{[X,Y]} Z = 0$, and similarly for the other combinations of X, Y and Z. Proving that the above must be zero and hence proving the claim.