Geometry Tutorial 11

Martijn Kluitenberg and Oscar Koster

7-2 2020

1. (Brachistochrone Problem.) *Imagine a metal bead with a wire threaded through a hole in it, so that the bead can slide with no friction along the wire. How can one choose the shape of the wire so that the time of descent under gravity (from rest) is smallest possible?*

   In this exercise, we will apply variational calculus to solve this famous problem. For convenience, choose coordinates such that the initial point is at the origin, the positive y-axis points downward (so the acceleration due to gravity is positive!) and the final point is at (a, b), with $a, b > 0$.

   (a) Draw a picture of the situation.

   Solution: Try this at home.

   (b) *(If you like physics)* Apply conservation of energy to show that the bead's velocity is $v = \sqrt{2gy}$.

   Solution: The kinetic energy is given by $\frac{1}{2}mv^2$, while the potential energy is $-mgy$. (Chosen to be zero at $t = 0$, and it should decrease with increasing y.) Hence, the total energy at any moment in time is zero, and we get

   $$\frac{1}{2}mv^2 = mgy \implies v(y) = \sqrt{2gy}.$$

   (c) Assume that the wire path can be written as a graph $x = x(y)$. Show that the travel time can be computed from the following integral:

   $$T = \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{(x'(y))^2 + 1}}{\sqrt{y}} \, \mathrm{d}y \, .$$

   Solution: Using that time equals distance over speed, we get

   $$\mathrm{d}t = \frac{\mathrm{d}s}{v} = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{v(y)} = \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}}{\sqrt{2gy}} \, \mathrm{d}y \, .$$

Integrating dt gives the total time taken, hence we get the desired result.

(d) Using $L(x, x', y) = \frac{\sqrt{(x')^2+1}}{\sqrt{y}}$, show that the Euler-Lagrange equation reduces to

$$\frac{(x')^2}{y(1+(x')^2)} = \text{constant}.$$

Solution:

$$\frac{\partial L}{\partial x'} = \frac{2x'}{2\sqrt{y}\sqrt{(x')^2+1}}$$
$$\frac{\partial L}{\partial x} = 0,$$

so the Euler-Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x} \implies \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{x'}{\sqrt{y}\sqrt{(x')^2+1}}\right) = 0$$

and hence, after taking the square,

$$\frac{(x')^2}{y(1+(x')^2)} = \text{constant}$$

(e) For convenience, we will set the constant above equal to $\frac{1}{2q}$. Resolve the above equation for x', and solve the resulting ODE using separation of variables.
Hint: Substitute $y = q(1-\cos\theta)$ for the integral.

Solution: We have

$$\frac{(x')^2}{y(1+(x')^2)} = \frac{1}{2q} \iff (2q-y)(x')^2 = y \iff x' = \sqrt{\frac{y}{y-2q}}.$$

So, integrating gives

$$
\begin{aligned}
x + C &= \int \sqrt{\frac{y}{2q - y}}\,\mathrm{d}y \\
&= q \int \sqrt{\frac{q(1-\cos\theta)}{2q - q(1-\cos\theta)}}\,\sin\theta\,\mathrm{d}\theta \\
&= q \int \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(\cos\theta - 1)(1-\cos\theta)}}\,\sin\theta\,\mathrm{d}\theta \\
&= q \int \frac{1-\cos\theta}{\sqrt{\sin^2\theta}}\,\mathrm{d}\theta \\
&= q(\theta - \sin\theta).
\end{aligned}
$$

Because $x(\theta = 0) = 0$, we can set the integration constant to zero.

(f) It follows from (d) that the solution to the Brachistochrone problem (in parametric form) can be written as

$$
\begin{cases}
x(\theta) = q(\theta - \sin\theta), \\
y(\theta) = q(1 - \cos\theta).
\end{cases}
$$

Show that these equations describe a *cycloid*, i.e. a curve obtained by rolling a circle of q over the x-axis, and tracing one point.

Solution: The parametric equations $(x(\theta), y(\theta))$ already follow from the previous part. The midpoint of the circle of radius q has coordinates $(q\theta, q)$, while the revolving point on the circle has coordinates $(-q\sin\theta, -q\cos\theta)$. (Draw a picture!) Adding these gives the parametric equations for the cycloid.

Remark: The cycloid has many more interesting properties. Oscillations of a ball rolling in a cycloid-shaped well are *isochronous*, meaning that the period is independent of the amplitude. You could think about what this fact implies for our Brachistochrone curve.