# Geometry Tutorial 11

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1. (Brachistochrone Problem.) *Imagine a metal bead with a wire threaded through a hole in it, so that the bead can slide with no friction along the wire. How can one choose the shape of the wire so that the time of descent under gravity (from rest) is smallest possible?*

   In this exercise, we will apply variational calculus to solve this famous problem. For convenience, choose coordinates such that the initial point is at the origin, the positive $y$-axis points downward (so the acceleration due to gravity is positive!) and the final point is at $(a, b)$, with $a, b > 0$.

   (a) Draw a picture of the situation.

   (b) *(If you like physics)* Apply conservation of energy to show that the bead's velocity is $v = \sqrt{2gy}$.

   (c) Assume that the wire path can be written as a graph $x = x(y)$. Show that the travel time can be computed from the following integral:

   $$T = \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{(x'(y))^2 + 1}}{\sqrt{y}} \, \mathrm{d}y \, .$$

   (d) Using $L(x, x', y) = \frac{\sqrt{(x')^2+1}}{\sqrt{y}}$, show that the Euler-Lagrange equation reduces to

   $$\frac{(x')^2}{y(1 + (x')^2)} = \text{constant}.$$

   (e) For convenience, we will set the constant above equal to $\frac{1}{2q}$. Resolve the above equation for $x'$, and solve the resulting ODE using separation of variables.
   *Hint:* Substitute $y = q(1 - \cos\theta)$ for the integral.

   (f) It follows from (d) that the solution to the Brachistochrone problem (in parametric form) can be written as

   $$\begin{cases} x(\theta) = q(\theta - \sin\theta), \\ y(\theta) = q(1 - \cos\theta). \end{cases}$$

   Show that these equations describe a *cycloid*, i.e. a curve obtained by rolling a circle of $q$ over the $x$-axis, and tracing one point.

   *Remark:* The cycloid has many more interesting properties. Oscillations of a ball rolling in a cycloid-shaped well are *isochronous*, meaning that the period is independent of the amplitude. You could think about what this fact implies for our Brachistochrone curve.