Geometry Tutorial 10

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1. Recall the definition of $\phi^* g$, for a given function $\phi : P \subset \mathbb{R}^n \to (Q, g)$. Why do we require that the derivative $\phi'(p)$ is injective for all $p \in P$? Does it matter if ϕ is injective?

Solution: We require that an inner product is non-degenerate, in the sense that

$$\phi^* g(p)(v, v) = 0 \implies v = 0.$$

Unpacking the definition, we get then

$$g(\phi(p))(\phi'(p)v, \phi'(p)v) = 0 \implies v = 0.$$

The metric g is assumed to be non-degenerate, so we do get $\phi'(p)v = 0$. However, this only implies that v is zero if we also assume that the derivative is injective. If not, then there is some nontrivial vector that gets mapped to zero, and our pull-back metric becomes degenerate.

It is not relevant if ϕ is injective or not. Indeed, you can prove that $\phi^* g$ is a metric, independent of injectivity of ϕ.

2. (Smooth invariance of dimension.) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a diffeomorphism (i.e. a C^1-bijection whose inverse $g : \mathbb{R}^m \to \mathbb{R}^n$ is also C^1.) In this exercise, we will show that in this case, $n = m$.

   (a) Prove that the derivative of $h = \mathrm{id}_{\mathbb{R}^n}$ is the identity map for all $p \in \mathbb{R}^n$.

   Solution: Since the Jacobian matrix is an $n \times n$ unit matrix (just take partial derivatives), the derivative map is the identity.

   Note that the basis is irrelevant here. With respect to any other basis, the derivative matrix becomes $X^{-1} I X = I$.

   (b) Prove that $f'(p)$ is an invertible linear map for all $p \in \mathbb{R}^n$. (*Hint:* Apply the chain rule to $f \circ g$ and $g \circ f$.)

Solution: The strategy here is to construct a two-sided inverse. Since $gf = \mathrm{id}_{\mathbb{R}^n}$, the chain rule at $p \in \mathbb{R}^n$ gives

$$(gf)'(p) = g'(f(p))f'(p).$$

Similarly, the chain rule at $f(p) \in \mathbb{R}^m$ gives

$$(fg)'(f(p)) = f'(g(f(p)))g'(f(p)) = f'(p)g'(f(p)),$$

where we used that $g(f(p)) = p$. We can denote $A = f'(p) : \mathbb{R}^n \to \mathbb{R}^m$ and $B = g'(f(p)) : \mathbb{R}^m \to \mathbb{R}^n$. Then, by part (a), the previous two equations read

$$I_{n\times n} = BA, \quad I_{m\times m} = AB,$$

thus $A = f'(p)$ is an invertible linear map with inverse B.

(c) Use linear algebra to conclude that $n = m$.

Solution: This can be seen in many different ways. Here, we will use the rank-nullity theorem. Since $f'(p)$ is injective, it has trivial kernel. Since it's surjective, the range is $\mathbb{R}^m$. Hence,

$$n = \dim(\mathbb{R}^n) = \dim(\ker A) + \dim(\mathrm{ran} A) = m,$$

and we are done.

Remark: This result also holds for a homeomorphism (so only continuity required!) $f : \mathbb{R}^n \to \mathbb{R}^m$, but it becomes much harder to prove.

3. Imagine a C^1 curve $\gamma : [a,b] \to P$ where (P,g) is a Riemannian chart and $\dot\gamma(t) \neq 0$ for all $t \in [a,b]$.

(a) Show that the length $L(\gamma)$ equals the length of any reparametrization $\gamma \circ j$ for $j : [a,b] \to [j(a), j(b)]$ any C^1 function with $j'(t) > 0$.

Solution: We can substitute $s = j(t)$ in the integral, noting that $|j'(t)| = j'(t)$, and applying the chain rule. This gives:

$$
\begin{aligned}
L(\gamma \circ j) &= \int_{j(a)}^{j(b)} \|(\gamma \circ j)'(t)\| \, \mathrm{d}t \\
&= \int_{j(a)}^{j(b)} \|\gamma'(j(t))j'(t)\| \, \mathrm{d}t \\
&= \int_{j(a)}^{j(b)} \|\gamma(j(t))\| j'(t) \, \mathrm{d}t \\
&= \int_{a}^{b} \|\dot{\gamma}(s)\| \, \mathrm{d}s \, , \\
&= L(\gamma)
\end{aligned}
$$

(b) Write down an integral for the the length of the curve along γ from $\gamma(a)$ to $\gamma(s)$.

Solution: We will denote the "arclength" function by α. By definition:

$$
\alpha(s) := \int_a^s \|\dot{\gamma}(t)\| \, \mathrm{d}t \, .
$$

(c) Show that we may reparametrize γ to obtain a new curve $\beta = \gamma \circ h$ such that $|\dot{\beta}(t)| = 1$ and $h'(t) > 0$ for all $t \in [a, b]$.

Solution: By the fundamental theorem of calculus, $\alpha'(s) = \|\dot{\gamma}(s)\| > 0$. Hence, by the inverse function theorem, $\alpha : [a, b] \to [0, L(\gamma)]$ has a differentiable inverse, which satisfies

$$
1 = (\alpha \circ \alpha^{-1})'(t) = \alpha'(\alpha^{-1}(t))(\alpha^{-1})'(t).
$$

Taking $h = \alpha^{-1}$, we get a curve $\beta : [0, L(\gamma)] \to P$, which has derivative

$$
\dot{\beta}(t) = (\gamma \circ h)'(t) = \dot{\gamma}(h(t))h'(t) = \dot{\gamma}(h(t)) \cdot \frac{1}{\alpha'(h(t))} = \dot{\gamma}(h(t)) \frac{1}{\|\dot{\gamma}(h(t))\|},
$$

and thus $\|\dot{\beta}(t)\| = 1$.

(d) Explain why the reparametrization of γ in the previous part is called a reparametrization by arc length.

Solution: The arclength along γ is used as the parameter.

4. Define the stereographic map $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\sigma(X,Y) = \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1} \right)$$

(a) Verify that σ is a bijection from $\mathbb{R}^2$ to $S^2 - \{(0,0,1)\}$.
Hint: Draw a picture, and try to guess a two-sided inverse.

Solution: Look up online what the stereographic projection does, and convince yourself that the inverse is given by

$$\tau : S^2 - \{(0,0,1)\} \to \mathbb{R}^2 : (x,y,z) \mapsto \tau(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

You should check carefully that $\tau\sigma = \mathrm{id}$ and that $\sigma\tau = \mathrm{id}$ (using that $x^2+y^2+z^2 = 1$, so x^2+y^2 can be replaced by $1-z^2$.)

(b) Define the metric $g = \sigma^* g_E$, the pull-back of the Euclidean metric along σ. Compute the coefficients of g with respect to the standard basis, i.e. g_{11}, g_{12}, g_{22}.
Hint: Maximally abuse the symmetry to avoid doing more calculations than necessary.

Solution: First, calculate the derivatives of σ w.r.t. X and Y. Because of some $X \leftrightarrow Y$ symmetries, you need to compute three of them. It turns out that $\sigma'(X,Y)$ has matrix

$$\frac{1}{(X^2+Y^2+1)^2} \begin{pmatrix} 2(-X^2+Y^2+1) & -4XY \\ -4XY & 2(X^2-Y^2+1) \\ 4X & 4Y \end{pmatrix}$$

Now, the coefficients of the pull-back metric are calculated by taking inner products between the columns of this matrix. For example, g_{11} is the inner product of the first column with itself. After carefully factorizing, you should get

$$g_{11} = g_{22} = \frac{4}{(X^2+Y^2+1)^2},$$

and $g_{12} = g_{21} = 0$.

(c) Calculate the length of the positive X-axis in the Riemannian chart $(\mathbb{R}^2, g)$.

Solution: Taking $\gamma(t) = (t,0)$ as a parametrization, we get $\dot\gamma(t) = e_1$, so

$$\|\dot\gamma(t)\|^2 = g_{11}(\gamma(t)) = \frac{4}{((X(t))^2+(Y(t))^2+1)^2} = \frac{4}{(t^2+1)^2}.$$

Hence,

$$L(\gamma) = \int_0^\infty \frac{2}{t^2+1}\,\mathrm{d}t = \left[2\tan^{-1}(t)\right]_0^\infty = \pi,$$

which is half the circumference of a great-circle.

Remark: In part (b), you should find that $g = f(X,Y)g_F$, where g_F is the standard Euclidean inner product on $\mathbb{R}^2$. How can you interpret this result?