Geometry Tutorial 10

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1. For an open set $P \subset \mathbb{R}^6$ defined by $x_1 \neq 0$ define the function $f : P \xmapsto{f} \mathbb{R}^4$ by $f(x) = (\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}, \frac{x_5}{x_1})$ Is f injective? What about $f'(1,1,1,1,1,1) : \mathbb{R}^6 \to \mathbb{R}^4$?

2. Pull back the standard Euclidean (Riemannian) metric on $\mathbb{R}^3$ using $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(x,y) = (x, y, |(x,y)|^2)$.
   (a) Does this define a Riemann metric on $\mathbb{R}^2$?
   (b) Compute the circumference of the circle with radius $r > 0$ in the Riemannian chart $(\mathbb{R}^2 - \{0\}, \varphi^* g_E)$.
   (c) How does your answer compare to the usual $2\pi r$?

3. Consider the hyperbolic half-plane $\mathbb{H}^2$ with metric $g(p)(v,w) = \frac{1}{y^2}\langle v, w\rangle_E$.
   (a) Compute the length of the straight line connecting $A(0,1)$ and $B(2,1)$.
   (b) Consider the circle with center $M(1,0)$ passing through A and B. Calculate the length of the arc between A and B. Compare your result with part (a). *Hint:* Use software to evaluate the integral.

4. Recall the definition of $\phi^* g$, for a given function $\phi : P \subset \mathbb{R}^n \to (Q, g)$. Why do we require that the derivative $\phi'(p)$ is injective for all $p \in P$? Does it matter if ϕ is injective?

5. (Smooth invariance of dimension.) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a diffeomorphism (i.e. a C^1-bijection whose inverse $g : \mathbb{R}^m \to \mathbb{R}^n$ is also C^1.) In this exercise, we will show that in this case, $n = m$.
   (a) Prove that the derivative of $h = \mathrm{id}_{\mathbb{R}^n}$ is the identity map for all $p \in \mathbb{R}^n$.
   (b) Prove that $f'(p)$ is an invertible linear map for all $p \in \mathbb{R}^n$. (*Hint:* Apply the chain rule to $f \circ g$ and $g \circ f$.)
   (c) Use linear algebra to conclude that $n = m$.

   Remark: This result also holds for a homeomorphism (so only continuity required!) $f : \mathbb{R}^n \to \mathbb{R}^m$, but it becomes much harder to prove.

6. Imagine a C^1 curve $\gamma : [a,b] \to P$ where (P, g) is a Riemannian chart and $\dot\gamma(t) \neq 0$ for all $t \in [a,b]$.

(a) Show that the length $L(\gamma)$ equals the length of any reparametrization $\gamma \circ j$ for $j : [a, b] \to [j(a), j(b)]$ any C^1 function with $j'(t) > 0$.

(b) Write down an integral for the the length of the curve along γ from $\gamma(a)$ to $\gamma(s)$.

(c) Show that we may reparametrize γ to obtain a new curve $\beta = \gamma \circ h$ such that $|\dot{\beta}(t)| = 1$ and $h'(t) > 0$ for all $t \in [a, b]$.

(d) Explain why the reparametrization of γ in the previous part is called a reparametrization by arc length.

7. Define the stereographic map $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\sigma(X, Y) = \left(\frac{2X}{X^2 + Y^2 + 1}, \frac{2Y}{X^2 + Y^2 + 1}, \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1} \right)$$

(a) Verify that σ is a bijection from $\mathbb{R}^2$ to $S^2 - \{(0, 0, 1)\}$.
Hint: Draw a picture, and try to guess a two-sided inverse.

(b) Define the metric $g = \sigma^* g_E$, the pull-back of the Euclidean metric along σ. Compute the coefficients of g with respect to the standard basis, i.e. g_{11}, g_{12}, g_{22}.
Hint: Maximally abuse the symmetry to avoid doing more calculations than necessary.

(c) Calculate the length of the positive X-axis in the Riemannian chart $(\mathbb{R}^2, g)$.

Remark: In part (b), you should find that $g = f(X, Y) g_F$, where g_F is the standard Euclidean inner product on $\mathbb{R}^2$. How can you interpret this result?

8. (Brachistochrone Problem.) *Imagine a metal bead with a wire threaded through a hole in it, so that the bead can slide with no friction along the wire. How can one choose the shape of the wire so that the time of descent under gravity (from rest) is smallest possible?*

In this exercise, we will apply variational calculus to solve this famous problem. For convenience, choose coordinates such that the initial point is at the origin, the positive y-axis points downward (so the acceleration due to gravity is positive!) and the final point is at (a, b), with $a, b > 0$.

(a) Draw a picture of the situation.

(b) *(If you like physics)* Apply conservation of energy to show that the bead's velocity is $v = \sqrt{2gy}$.

(c) Assume that the wire path can be written as a graph $x = x(y)$. Show that the travel time can be computed from the following integral:

$$T = \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{(x'(y))^2 + 1}}{\sqrt{y}} \, \mathrm{d}y \, .$$

(d) Using $L(x, x', y) = \frac{\sqrt{(x')^2+1}}{\sqrt{y}}$, show that the Euler-Lagrange equation reduces to

$$\frac{(x')^2}{y(1 + (x')^2)} = \text{constant.}$$

(e) For convenience, we will set the constant above equal to $\frac{1}{2q}$. Resolve the above equation for x', and solve the resulting ODE using separation of variables.
Hint: Substitute $y = q(1-\cos\theta)$ for the integral.

(f) It follows from (d) that the solution to the Brachistochrone problem (in parametric form) can be written as

$$\begin{cases} x(\theta) = q(\theta - \sin\theta), \\ y(\theta) = q(1-\cos\theta). \end{cases}$$

Show that these equations describe a *cycloid*, i.e. a curve obtained by rolling a circle of q over the x-axis, and tracing one point.

Remark: The cycloid has many more interesting properties. Oscillations of a ball rolling in a cycloid-shaped well are *isochronous*, meaning that the period is independent of the amplitude. You could think about what this fact implies for our Brachistochrone curve.

Geodesics and Curvature

9. Describe qualitatively the geodesics on:

   1. A spheroid (the surface obtained by rotating an ellipse around one of its axes).
   2. A cylinder.
   3. A torus.
   4. A cone

10. The shortest path between two points may not always take you along a geodesic. For example prove that in the Riemannian chart (P, g_E) with $P = \mathbb{R}^2 - \{0\}$ and $g_E(p)$ the standard Euclidean inner product at all points there does NOT exist a geodesic connecting $-e_1$ to e_1.

11. Minimize the length of a curve in the Euclidean plane directly, do you still find straight lines as solutions?

12. Show that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Why is this useful for calculating Christoffel symbols?

13. (a) Compute the Christoffel symbols of the sphere in spherical coordinates.
    (b) Show that the great circles that are meridians are geodesics on the sphere.
    (c) Explain why moving from point $p = (1,0,0)$ to $q = (0,1,0)$ along a great circle may not be the shortest path from p to q on the sphere.
    (d) Why does this actually prove that all great circles on the sphere are geodesics?

14. (a) For vector fields $X, Y, Z : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $X(x,y,z) = (x,0,0)$, $Y(x,y,z) = (x,y,z)$ and $Z(x,y,z) = e_2 + xe_3$ compute $H = \partial_X Y$ and $\partial_Z H$. Also compute $[X,[Y,Z]]$.
    (b) Verify the Jacobi identity in the above example

    $$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

    (c) Prove the Jacobi identity for any vector fields X, Y, Z on $\mathbb{R}^n$.