Geometry: Tutorial I

Martijn Kluitenberg and Oscar Koster

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1. Can you give an example of an affine space that is not a vector space?

> **Solution:** Any plane or line not crossing the origin in $\mathbb{R}^n$. Both are affine because they satisfy the definition. They are not vector spaces because suppose L is a line, for any $a, b \in L$ we don't necessarily have $a - b \in L$ (for instance $a - a \notin L$) because we have no fixed origin.

2. Let $\mathscr{F}$ be a subset of an affine space $\mathscr{E}$, directed by the vector space E and let $A \in \mathscr{F}$ be a point such that $\Theta_A(\mathscr{F}) =: F$ is a linear subspace of E. Prove that $\Theta_B(\mathscr{F}) = F$ for *any* point $B \in \mathscr{F}$.

> **Solution:** The problem here is that we don't know a priori that $\Theta_B(\mathscr{F})$ is a linear subspace of E. To solve this, we will use that the transition map
>
> $$\Theta_A \circ \Theta_B^{-1}(v) = v + \overrightarrow{AB}$$
>
> is a translation (see lecture notes). We obtain
>
> $$\Theta_B(\mathscr{F}) = \Theta_B \circ \Theta_A^{-1} \circ \Theta_A(\mathscr{F}) = \overrightarrow{BA} + \Theta_A(\mathscr{F}) = \Theta_A(\mathscr{F}),$$
>
> where we have inserted the identity map, and used that $F = \Theta_A(\mathscr{F})$ is a linear subspace, so that $v + F = F$ for any $v \in F$.

3. Prove that through any two distinct points of an affine space $\mathscr{E}$ passes a unique line.

> **Solution:** Note that $\langle A, B \rangle$ is an affine subspace of $\mathscr{E}$ containing A and B. This space is equal to the affine subspace of $\mathscr{E}$ through A which is directed by $\operatorname{Span} \overrightarrow{AB} \neq 0$. In particular, it is a line through A and B.

Let $\mathscr{D}$ be any line through A and B, i.e. a one-dimensional affine subspace of $\mathscr{E}$ containing A and B. Obviously, $\langle A, B\rangle \subseteq \mathscr{D}$. (It is the smallest such space.)

Conversely, let D be the direction of $\mathscr{D}$, and let $C \in \mathscr{D}$ arbitrary. Then, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors in the one-dimensional space D, and hence $\overrightarrow{AC} = \lambda\overrightarrow{AB}$ for some $\lambda \in \mathbb{K}$. It follows that $\overrightarrow{AC} \in \operatorname{Span}\overrightarrow{AB}$, so that $C \in \langle A, B\rangle$.

4. Let A be an $m \times n$ matrix, and let $b \in \mathbb{K}^m$. Define

$$\mathscr{F} = \{x \in \mathbb{K}^n : Ax = b\}.$$

Prove that $\mathscr{F}$ is an affine subspace of $\mathbb{K}^m$. What is its direction? When it it empty? Express the dimension of $\mathscr{F}$ in terms of the rank of A.

Solution: If $\mathscr{F} = \emptyset$, then it is clearly an affine subspace of $\mathbb{K}^m$. Suppose that $\mathscr{F} \neq \emptyset$, i.e. that $Ax = b$ has at least one solution (say x_0), i.e. b is in the column space of A. We claim that $x \in \mathbb{K}^n$ solves $Ax = b$ if and only if $x = x_0 + u$ for some $u \in \ker A$, which in turn implies that $\mathscr{F}$ is an affine subspace of $\mathbb{K}^m$, directed by $\ker A$.

The claim above is a standard result from linear algebra, but let's prove it nonetheless. If $x = x_0 + u$, then obviously $Ax = Ax_0 + Au = b + 0 = b$. Conversely, if $Ax = b$, then $x - x_0 \in \ker A$, so $x = x_0 + (x - x_0)$ has the required form.

Finally, the rank-nullity theorem implies that $\dim \mathscr{F} = n - r$, where r is the rank of A.

5. Let $\mathscr{E}$ and $\mathscr{F}$ be affine spaces, directed by E and F respectively, and let $\phi : \mathscr{E} \to \mathscr{F}$ be an affine mapping, and let $\mathscr{M}$ be an affine subspace of $\mathscr{F}$. Prove that $\phi^{-1}(\mathscr{M})$ is an affine subspace of $\mathscr{E}$.

Solution: If $\mathscr{M} = \emptyset$, then the result is immediate. So, we can let $O \in \phi^{-1}(\mathscr{M})$, i.e. $\phi(O) \in \mathscr{M}$. Since ϕ is affine, there is a linear map $f : E \to M$ such that $f(\overrightarrow{OA}) = \overrightarrow{\phi(O)\phi(A)}$ for all $A \in \mathscr{E}$. Since $\mathscr{M}$ is an affine subspace, $M := \Theta_{\phi(O)}\mathscr{M}$ is a linear subspace of F. Note that $f^{-1}(M)$ is thus a linear subspace of E. We claim that $\Theta_O(\phi^{-1}(\mathscr{M})) = f^{-1}(M)$, and hence that $\phi^{-1}(\mathscr{M})$ is an affine subspace of $\mathscr{E}$.

($\subseteq$) Suppose that $\overrightarrow{OA} \in \Theta_O(\phi^{-1}(\mathscr{M}))$, where $\phi(A) \in \mathscr{M}$. We have that

$$f(\overrightarrow{OA}) = \overrightarrow{\phi(O)\phi(A)} \in \Theta_{\phi(O)}(\mathscr{M}).$$

($\supseteq$) Suppose that $v = \overrightarrow{OA} \in f^{-1}(M)$, i.e. $f(v) \in \Theta_{\phi(O)}(\mathscr{M})$. Hence, $f(v) = \overrightarrow{\phi(O)B}$ for some $B \in \mathscr{M}$. But also

$$f(v) = f(\overrightarrow{OA}) = \overrightarrow{\phi(O)\phi(A)},$$

so we obtain $B = \phi(A)$, and thus $v = \Theta_O(A)$, with $A \in \phi^{-1}(\mathscr{M})$.

6. Prove that an affine mapping is completely determined by the image of an affine frame.

Solution: Let $(A_0, A_1, \ldots, A_n)$ be an affine frame of the affine space $\mathscr{E}$, directed by E. Note that $\{A_0A_1, \ldots, A_0A_n\}$ is a basis for E. Let $\phi : \mathscr{E} \to \mathscr{F}$ be an affine map, then by definition, for some $O \in \mathscr{E}$ there is a linear map $f : E \to F$ such that for all $M \in \mathscr{E}$ we have $\overrightarrow{\phi(O)\phi(M)} = f(\overrightarrow{OM})$. We can rewrite the vectors by means of the given basis to get $\overrightarrow{\phi(O)\phi(M)} = f(\overrightarrow{OM}) = f(a_0\overrightarrow{A_0A_1} + \cdots + a_n\overrightarrow{A_0A_n}) = a_0 f(\overrightarrow{A_0A_1}) + \cdots + a_n f(\overrightarrow{A_0A_n}) = a_0\overrightarrow{\phi(A_0)\phi(A_1)} + \cdots + a_n\overrightarrow{\phi(A_0)\phi(A_n)}$, where we use the linearity of f. The coefficients are unique. This shows that an affine map is completely determined by the images of the affine frame.