Geometry Homework 1

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You can receive $11 + 8 + 5 + 6 + 8 = 38$ points. Your grade is computed $(\frac{\text{number of points}}{38}) * 9 + 1$.

1. (a) Let V_1 and V_2 be linear subspaces of a vector space V. Under which conditions is $V_1 \cup V_2$ a linear subspace of V? Prove your claim.

Solution: Claim: $V_1 \cup V_2$ is a linear subspace of V if and only if $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.
(1 point)

$(\Longleftarrow)$ If $V_1 \subseteq V_2$, then $V_1 \cup V_2 = V_2$, which is clearly a linear subspace. The case $V_2 \subseteq V_1$ is identical.
(1 point)

$(\Longrightarrow)$ Suppose that $V_1 \cup V_2$ is a linear subspace, but that there exist $v_1 \in V_1 \setminus V_2$ and $v_2 \in V_2 \setminus V_1$. Since both v_1 and v_2 are in $V_1 \cup V_2$, so is their difference.
(1 point)

But now $v_2 = v_1 + (v_2 - v_1)$, so $v_2 - v_1$ can't be in V_1. Also. $v_1 = v_2 - (v_2 - v_1)$, so it can't be in V_2 either. This gives a contradiction.
(2 points)

(b) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be affine subspaces of an affine space $\mathcal{E}$, directed by E. Under which condition is $\mathcal{F}_1 \cup \mathcal{F}_2$ an affine subspace of $\mathcal{E}$? Prove your claim.

Solution: Claim: $\mathcal{F}_1 \cup \mathcal{F}_2$ an affine subspace of $\mathcal{E}$ if and only if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ or $\mathcal{F}_2 \subseteq \mathcal{F}_1$.
(1 point)

$(\Longleftarrow)$ This is the same proof as in part (a).

$(\Longrightarrow)$ If either $\mathcal{F}_1$ or $\mathcal{F}_2$ is empty, the result certainly holds. So, we can assume from now on that both are nonempty.

(1 point)

If $\mathcal{F}_1 \cap \mathcal{F}_2$ is nonempty, we can take an element A of the intersection. In this case, both $\Theta_A(\mathcal{F}_1)$ and $\Theta_A(\mathcal{F}_2)$ are linear subspaces of E. Their union $\Theta_A(\mathcal{F}_1 \cup \mathcal{F}_2) = \Theta_A(\mathcal{F}_1) \cup \Theta_A(\mathcal{F}_2)$ is a linear subspace of E if and only if one of them is contained in the other. Hence, $\mathcal{F}_1 \subseteq \mathcal{F}_2$ or the other way around.
(2 points)

Finally, note that $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ is not possible. Indeed, if $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, and $\Theta_A(\mathcal{F}_1 \cup \mathcal{F}_2)$ is a linear subspace of E, we get a contradiction. You can use linear algebra to convince yourself that the line segment $[A, B]$ has to meet $\mathcal{F}_1 \cap \mathcal{F}_2$ somewhere. This completes the proof.
(2 points)

2. Let A be a point on $\mathcal{D}$ be a line in an affine plane $\mathcal{P}$ with underlying vector space P. For a one-dimensional linear subspace $L \subset P$ not equal to $\Theta_A(\mathcal{D})$ we consider the mapping $\pi : \mathcal{P} \to \mathcal{P}$ defined as follows. $\pi(M)$ is the point M' defined by $M' \in \mathcal{D}$ and $\overrightarrow{MM'} \in L$.

   (a) Why can there be only one point M' with properties $M' \in \mathcal{D}$ and $\overrightarrow{MM'} \in L$?

   Solution: Note that by assumption we have that that $L \neq \Theta_A(\mathcal{D})$, meaning the lines are not parallel. This means that MM' and $\mathcal{D}$ are not parallel lines.
   (1 point)

   Two non-parallel lines have a unique intersection. Therefore, there is a unique intersection between L and $\mathcal{D}$ given by M'.
   (1 point)

   (b) Prove that π is an affine map.

   Solution:
   π is an affine map if for $O \in \mathcal{P}$ there is a linear map $\overrightarrow{\pi} : P \to P$ such that for every $M \in \mathcal{P}$ we have $\overrightarrow{\pi(O)\pi(M)} = \overrightarrow{\pi}(\overrightarrow{OM})$.
   (1 point)

   Pick A to be the origin of $\mathcal{P}$ i.e. $O = A$. Note that $\pi(A) = A$, so $\overrightarrow{\pi(A)\pi(M)} = \overrightarrow{AM'}$. Set the map $\overrightarrow{\pi} = AM'$ in order to satisfy the definition. We still need to show that $\overrightarrow{\pi}$ is a linear map. First, let $M, N \in \mathcal{P}$ then $\overrightarrow{\pi}(\overrightarrow{A(M+N)}) = \overrightarrow{A(M+N)'} = \overrightarrow{AM'} + \overrightarrow{AN'} = \overrightarrow{\pi}(\overrightarrow{AM}) + \overrightarrow{\pi}(\overrightarrow{AN})$. Secondly, let c be a scalar in $\mathbb{K}$ and $M \in \mathcal{P}$ then $\overrightarrow{\pi}(\overrightarrow{cAM}) = \overrightarrow{cAM'} = c\overrightarrow{AM'} = c\overrightarrow{\pi}(\overrightarrow{AM})$. Proving that $\overrightarrow{\pi}$ is a linear map.
   (2 points)

(c) What is the linear map $\overrightarrow{\pi} : P \to P$ associated to π?

Solution: The linear map $\overrightarrow{\pi} : P \to P$ associated to π is given by $\overrightarrow{\pi}(\overrightarrow{OM}) = \overrightarrow{OM'}$ for a point $M \in \mathcal{P}$ and an origin $O \in \mathcal{P}$.
(1 point)

(d) Why is π not defined if L is in the same direction as $\mathcal{D}$?

Solution: Suppose L is in the same direction as $\mathcal{D}$, then L would be either parallel to $\mathcal{D}$ or $L = \mathcal{D}$. In the case L is parallel to $\mathcal{D}$ there is no intersection M' between between L and $\mathcal{D}$, which means the map π is not defined.
(1 point)

In the case $\mathcal{D} = L$ we would have an infinite number of intersections M', this means the map π is not well-defined.
(1 point)

3. Imagine a 2-simplex $[A_0, A_1, A_2]$ with centroid G and G_i the centroid of the face opposite to A_i. Using $\overrightarrow{GA_0} + \overrightarrow{GA_1} + \overrightarrow{GA_2} = 0 = \overrightarrow{G_0A_1} + \overrightarrow{G_0A_2}$, prove that $2\overrightarrow{GG_0} = \overrightarrow{A_0G}$.

Solution: Note that we can write $\overrightarrow{GG_0} = \overrightarrow{GA_1} + \overrightarrow{A_1G_0}$ and similarly as $\overrightarrow{GG_0} = \overrightarrow{GA_2} + \overrightarrow{A_2G_0}$.
(2 points)

Adding these together we get that $2\overrightarrow{GG_0} = \overrightarrow{GA_1} + \overrightarrow{A_1G_0} + \overrightarrow{GA_2} + \overrightarrow{A_2G_0} = \overrightarrow{GA_1} - \overrightarrow{G_0A_1} + \overrightarrow{GA_2} - \overrightarrow{G_0A_2}$. Rewriting $\overrightarrow{GA_0} + \overrightarrow{GA_1} + \overrightarrow{GA_2} = 0 = \overrightarrow{G_0A_1} + \overrightarrow{G_0A_2}$ we get that $\overrightarrow{GA_1} - \overrightarrow{G_0A_1} + \overrightarrow{GA_2} - \overrightarrow{G_0A_2} = -\overrightarrow{GA_0} = \overrightarrow{A_0G}$ proving the claim.
(3 point)

4. Prove that for any nonzero $\lambda \in \mathbb{R}$ and $O \in \mathcal{E}$ the dilation $h_{O,\lambda}$ sends any n-simplex in $\mathcal{E}$ to another n-simplex in $\mathcal{E}$.

Solution: Let $[A_0, \ldots, A_n]$ be an n-simplex in $\mathcal{E}$. We want to show that $\{h(A_0), \ldots, h(A_n)\}$ still spans an n-dimensional space, and that $[h(A_0), \ldots, h(A_n)] = h([A_0, \ldots, A_n])$.
(1 points)

Since the points $A_0, \ldots, A_n$ are affine independent, we have that $\text{Span}\{\overrightarrow{A_0A_i} : i = 1, \ldots, n\}$ is an n-dimensional linear subspace of E. Since h is an affine map, $\overrightarrow{h(A_0)h(A_i)} = \lambda\overrightarrow{A_0A_i}$. Then, using $\lambda \neq 0$, we see that

$$\text{Span}\{\overrightarrow{h(A_0)h(A_i)} : i = 1, \ldots, n\} = \text{Span}\{\lambda\overrightarrow{A_0A_i} : i = 1, \ldots, n\},$$

so $[h(A_0), \ldots, h(A_n)]$ is an n simplex.
(2 points)

($\subseteq$) If $B \in [h(A_0), \ldots, h(A_n)]$, then there are $\beta_i \in [0,1]$ such that $\sum_{i=0}^{n} \beta_i \overrightarrow{Bh(A_i)} = 0$. Since h is bijective, we can choose $A \in \mathcal{E}$ s.t. $h(A) = B$. We have that

$$0 = \sum_{i=0}^{n} \beta_i \overrightarrow{h(A)h(A_i)} = \lambda \sum_{i=0}^{n} \beta_i \overrightarrow{AA_i},$$

from which it follows that $A \in [A_0, \ldots, A_n]$. Thus, $B = h(A) \in h([A_0, \ldots, A_n])$.
(2 points)

($\supseteq$) If $B = h(A)$, with $A \in [A_0, \ldots, A_n]$, then by the same computation as above, we have that $B \in [h(A_0), \ldots, h(A_n)]$. So, we are done.
(1 points)

5. By a rotation around point A in affine Euclidean plane $\mathcal{E}$ we mean any composition of two affine reflections in lines passing through A. For $A, B \in \mathcal{E}$ let ρ_A be a rotation around A and ρ_B a rotation around $B \neq A$.

   (a) What is the determinant of the linear map $\overrightarrow{\rho_A} : E \to E$ associated to ρ_A?

   Solution: We know the determinant of a linear reflection is -1. A rotation around a point $A \in \mathcal{E}$, ϕ_A is the composition of two reflections of lines $\mathcal{F}$ and cF' crossing A. This implies that for a rotation ρ_A we have $\det(\rho) = \det(\sigma_{\mathcal{F}} \circ \sigma_{\mathcal{F}'}) = \det(\sigma_{\mathcal{F}}) \det(\sigma_{\mathcal{F}'} = -1 \cdot -1 = 1$.
   (2 points)

   (b) Prove that $\overrightarrow{\rho_A \circ \rho_B} = \overrightarrow{\rho_A} \circ \overrightarrow{\rho_B}$.

   Solution: Choose an arbitrary origin $O \in \mathcal{E}$ and $M \in \mathcal{E}$ then:

$$\begin{aligned} \overrightarrow{O\rho_A \circ \rho_B(M)} &= \overrightarrow{\rho_A}(\overrightarrow{O\rho_B(M)}) \\ &= \overrightarrow{\rho_A}\overrightarrow{\rho_B}(\overrightarrow{OM}). \end{aligned}$$

   Since we have picked an arbitrary O and M in $\mathcal{E}$ we have found that $\overrightarrow{\rho_A \circ \rho_B} = \overrightarrow{\rho_A} \circ \overrightarrow{\rho_B}$.
   (2 points)

   (c) What is the determinant of $\overrightarrow{\rho_A \circ \rho_B}$?

Solution: Use question (b) to see that $\overrightarrow{\rho_A \circ \rho_B} = \overrightarrow{\rho_A} \circ \overrightarrow{\rho_B}$ and hence $\det(\overrightarrow{\rho_A} \circ \overrightarrow{\rho_B}) = \det(\rho_A)\det(\rho_B) = 1 \cdot 1 = 1$.
(1 point)

(d) Is it true that $\rho_A \circ \rho_B$ must also be a rotation around some point $C \in \mathcal{E}$? Prove or provide a counterexample.

Solution: Consider the rotations around point A by θ denoted $\rho_{A,\theta}$ and around point B by $-\theta$ denoted by $\rho_{B,-\theta}$.
(2 points)

This map is a translation and not a rotation around a point.
(1 point)