Geometry Homework 3

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You can receive $18 + 5 + 10 = 33$ points. Your grade is computed $(\frac{\text{number of points}}{35}) * 9 + 1$.

1. Consider the surface Σ_f obtained by rotating the graph of a positive C^2 function $f : [-1, 1] \to \mathbb{R}$ around the x-axes. Inspired by playing with soap films spanned between two circles we would like to find a f such that Σ_f has minimal area. Recall the area of Σ is given by the integral $2\pi \int_{-1}^{1} f(t)\sqrt{1 + \dot{f}(t)^2}dt$.

   (a) **(4 points)** Write down a differential equation that functions f minimizing the area of Σ_f should satisfy.

Solution:

Let $S(f) = \text{Area}(\Sigma_f) = 2\pi \int_{-1}^{1} f(t)\sqrt{1 + \dot{f}(t)^2}$. Using theorem 6.1 of the lecture notes, we have that the smooth function f is a minimum of S if it satisfies the Euler-Lagrange equations. Let us define $\mathcal{L}(f(t), \dot{f}(t), t) = f(t)\sqrt{1 + \dot{f}(t)^2}$. For $n = 1$ the Euler-Lagrange equations states:

$$\frac{\partial \mathcal{L}(f(t), \dot{f}(t), t)}{\partial f(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}(f(t), \dot{f}(t), t)}{\partial \dot{f}(t)}.$$

The left-hand side is given by:

$$\frac{\partial \mathcal{L}(f(t), \dot{f}(t), t)}{\partial f(t)} = \sqrt{1 + \dot{f}(t)^2}. \tag{1}$$

(1 point)
For the right-hand side, we first notice that by the chain rule:

$$\frac{\partial \mathcal{L}(f(t), \dot{f}(t), t)}{\partial \dot{f}(t)} = f(t)\frac{\dot{f}(t)}{\sqrt{1 + \dot{f}(t)^2}}.$$

After which we can take a derivative w.r.t. t using the quotient rule and the chain rule for differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}(f(t),\dot{f}(t),t)}{\partial \dot{f}(t)} = \frac{\left(\dot{f}(t)^2 + f(t)\ddot{f}(t)\right)\sqrt{1+\dot{f}(t)^2} - f(t)\dot{f}(t)\left(\frac{\dot{f}(t)\ddot{f}(t)}{\sqrt{1+\dot{f}(t)^2}}\right)}{1+\dot{f}(t)^2} \tag{2}$$

(2 points

Setting (2) equal to (1), we obtain:

$$\begin{aligned}
\frac{\left(\dot{f}(t)^2 + f(t)\ddot{f}(t)\right)\sqrt{1+\dot{f}(t)^2} - f(t)\dot{f}(t)\left(\frac{\dot{f}(t)\ddot{f}(t)}{\sqrt{1+\dot{f}(t)^2}}\right)}{1+\dot{f}(t)^2} &= \sqrt{1+\dot{f}(t)^2} \\
\Rightarrow \left(\dot{f}(t)^2 + f(t)\ddot{f}(t)\right)\sqrt{1+\dot{f}(t)^2} - f(t)\dot{f}(t)\left(\frac{\dot{f}(t)\ddot{f}(t)}{\sqrt{1+\dot{f}(t)^2}}\right) &= \left(1+\dot{f}(t)^2\right)^{3/2} \\
\Rightarrow \dot{f}(t)^2 + f(t)\ddot{f}(t) - f(t)\dot{f}(t)\left(\frac{\dot{f}(t)\ddot{f}(t)}{1+\dot{f}(t)^2}\right) &= 1+\dot{f}(t)^2 \\
\Rightarrow f(t)\ddot{f}(t) &= 1 + \frac{f(t)\dot{f}(t)^2\ddot{f}(t)}{1+\dot{f}(t)^2}.
\end{aligned}$$

Which is a differential equation that functions f minimizes the area of Σ_f should satisfy. **(1 point)**.

(b) **(2 points)** Verify that $f(t) = \cosh(t)$ solves the equation you found.

Solution: Notice:

$$\begin{aligned}
\dot{f}(t) &= \sinh(t) \\
\ddot{f}(t) &= \cosh(t) \\
\cosh^2(t) - \sinh^2(t) &= 1.
\end{aligned}$$

Plugging the above into the differential equation found in (a) we get:

$$
\begin{aligned}
\cosh(t)^2 &= 1 + \frac{\cosh^2 t \sinh^2 t}{1+\sinh^2(t)} \\
&= 1 + \frac{\cosh^2 t \sinh^2 t}{\cosh^2(t)} \\
&= 1 + \sinh^2(t) \\
&= \cosh^2(t).
\end{aligned}
$$

Therefore, the differential equation is satisfied by $f(t) = \cosh(t)$. **(2 points)**

(c) **(2 points)** Setting $P = (-1,1) \times (0, 2\pi)$ we get the Riemannian chart (P, g) of Σ_f using $\phi : P \to \mathbb{R}^3$ defined by $\phi(t,\theta) = (t, f(t)\cos\theta, f(t)\sin\theta)$ and with $g = \phi^* g_E$ the pull-back metric. Give a formula for the function $g_{1,2} : P \to \mathbb{R}$ in terms of f.

Solution: First, compute the Jacobian of the map ϕ

$$
\phi'(t,\theta) = \begin{pmatrix} 1 & 0 \\ \dot{f}(t)\cos\theta & -f(t)\sin\theta \\ \dot{f}(t)\sin\theta & f(t)\cos\theta \end{pmatrix}.
$$

Then the pull-back metric can be computed as:

$$
\begin{aligned}
g_{1,2} &= \phi^* g_E(e_1, e_2) \\
&= \left\langle \begin{matrix} 1 & 0 \\ \dot{f}(t)\cos\theta & -f(t)\sin\theta \\ \dot{f}(t)\sin\theta & f(t)\cos\theta \end{matrix} \right\rangle \\
&= 1 \cdot 0 - f(t)\dot{f}(t)\cos\theta\sin\theta + f(t)\dot{f}(t)\sin theta\cos\theta \\
&= 0.
\end{aligned}
$$

(2 points)

(d) **(7 points)** For constant c we consider the curves μ, λ in P given by $\mu(t) = (t, c)$ and $\lambda(t) = (c, t)$. Give an example of f such that all the μ, λ are geodesics on Σ_f.

Solution: Notice that $g_{1,2} = g_{2,1}$, which were already computed in question (c). We compute in similar ways:

$$
\begin{aligned}
g_{1,1} &= 1 + \dot{f}(t)^2 \\
g_{2,2} &= f(t)^2.
\end{aligned}
$$

Which means we can write g and g^{-1}, for $i, j \in 1, 2$, in the following matrices:

$$g_{i,j} = \begin{pmatrix} 1+\dot{f}(t)^2 & 0 \\ 0 & f(t)^2 \end{pmatrix} \qquad \text{and } g^{-1}_{i,j} = \begin{pmatrix} \frac{1}{1+\dot{f}(t)^2} & 0 \\ 0 & \frac{1}{f(t)^2} \end{pmatrix}.$$

(1 point)
We can now compute, by hand or using Mathematica, the Christoffel symbols as follows:

$$\Gamma^1_{1,1} = \frac{1}{2}\left(g^{-1}_{11}\left(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}\right)\right) = \frac{1}{2}\left(\frac{2\dot{f}(t)\ddot{f}(t)}{1+\dot{f}(t)^2}\right) = \frac{\dot{f}(t)\ddot{f}(t)}{1+\dot{f}(t)^2}$$

$$\Gamma^1_{1,2} = \frac{1}{2}\left(g^{-1}_{11}\left(\frac{\partial g_{21}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_1}\right)\right) = 0$$

$$\Gamma^1_{2,1} = \frac{1}{2}\left(g^{-1}_{11}\left(\frac{\partial g_{11}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_1} - \frac{\partial g_{12}}{\partial x_1}\right)\right) = 0$$

$$\Gamma^1_{2,2} = \frac{1}{2}\left(g^{-1}_{11}\left(\frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1}\right)\right) = \frac{1}{2}\left(\frac{-2f(t)\dot{f}(t)}{1+\dot{f}(t)^2}\right) = -\frac{f(t)\dot{f}(t)}{1+\dot{f}(t)^2}$$

$$\Gamma^2_{1,1} = \frac{1}{2}\left(g^{-1}_{22}\left(\frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2}\right)\right) = 0$$

$$\Gamma^2_{1,2} = \frac{1}{2}\left(g^{-1}_{22}\left(\frac{\partial g_{22}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_2}\right)\right) = \frac{1}{2}\left(\frac{2f(t)\dot{f}(t)}{f(t)^2}\right) = \frac{f(t)\dot{f}(t)}{f(t)^2} = \frac{\dot{f}(t)}{f(t)}$$

$$\Gamma^2_{2,1} = \frac{1}{2}\left(g^{-1}_{22}\left(\frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_2}\right)\right) = \frac{1}{2}\left(\frac{2f(t)\dot{f}(t)}{f(t)^2}\right) = \frac{f(t)\dot{f}(t)}{f(t)^2} = \frac{\dot{f}(t)}{f(t)}$$

$$\Gamma^2_{2,2} = \frac{1}{2}\left(g^{-1}_{22}\left(\frac{\partial g_{22}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_2}\right)\right) = 0$$

(2 points)
We have to determine when $\mu(t) = (t, c) \in \mathbb{R}^2$ and $\lambda = (c, t) \in \mathbb{R}^2$ are geodesics. Notice that the dimension of the space is $n = 2$, so we consider $i, j, r \in \{1, 2\}$. To do so we need to check when these curves satisfy the geodesic equation.
For $\mu(t) = (t, c)$ we obtain for $\mathbf{r = 1}$:

$$\ddot{\mu_1}(t) + \sum_{i,j=1}^{2} \Gamma^1_{ij}(\mu(t))\dot{\mu}_i(t)\dot{\mu}_j(t) = \frac{\dot{f}(t)\ddot{f}(t)}{1+\dot{f}(t)^2}.$$

This equation is zero if and only if $\dot{f}(t) = 0$ or $\ddot{f}(t) = 0$. And we obtain for $\mathbf{r = 2}$:

$$\ddot{\mu_2}(t) + \sum_{i,j=1}^{2} \Gamma^2_{ij}(\mu(t))\dot{\mu}_i(t)\dot{\mu}_j(t) = 0$$

Proving, that $\mu(t)$ is only a geodesic if and only if $\dot{f}(t) = 0$ or $\ddot{f}(t) = 0$.

Similarly, for $\lambda(t) = (c, t)$ we obtain for $\mathbf{r = 1}$:

$$\ddot{\lambda}_1(t) + \sum_{i,j=1}^{2} \Gamma^1_{ij}(\lambda(t))\dot{\lambda}_i(t)\dot{\lambda}_j(t) = -\frac{f(t)\dot{f}(t)}{1+\dot{f}(t)^2}.$$

Proving that $\lambda(t)$ is a geodesic if and only if $f(t) = 0$ or $\dot{f}(t) = 0$. **(2 points)** And for $\mathbf{r = 2}$:

$$\ddot{\lambda}_2(t) + \sum_{i,j=1}^{2} \Gamma^2_{ij}(\lambda(t))\dot{\lambda}_i(t)\dot{\lambda}_j(t) = 0.$$

Therefore, in order for both $\lambda(t)$ and $\mu(t)$ to be a geodesic we need to have that $\dot{f}(t) = 0$, therefore $f(t)$ needs to be any constant function. **(2 points)**

(e) **(1 point)** Give an example of f such that not all curves μ are geodesics on Σ_f.

Solution: If $\dot{f}(t)$ and $\ddot{f}(t)$ are both not zero, we have an example of a curve f such that not all curves $\mu(t)$ on f are geodesics. We can for example take the function $f(t) = t^4$, then for $t \neq 0$, $\dot{f}(t) = 4t^3 \neq 0$ and $\ddot{f}(t) = 12t^2 \neq 0$. Which shows that not all $\mu(t)$ are geodesics on f. **(1 point)**

(f) **(2 points)** Prove that ν is always a geodesic of Σ_f where ν is obtained from μ by reparametrizing it to have constant speed (with respect to g).

Solution: Consider a curve $\nu(\Gamma) = \mu(\Gamma) = (\Gamma, c)$ which has constant speed. Recall that the speed of a curve μ w.r.t. a metric g is given by:

$$|\dot{\mu}(\Gamma)| = \sqrt{g(\mu(\Gamma))(\dot{\mu}(\Gamma), \dot{\mu}(\Gamma))}.$$

Rewriting this using $\dot{\mu}(\Gamma) = e_1$ and the definition of the pull-back metric, we get:

$$|\dot{\mu}(\Gamma)| = \sqrt{g_{11}} = \sqrt{1 + \dot{f}(\Gamma)^2}.$$

Since this should be a constant function, we know that $\frac{\mathrm{d}}{\mathrm{d}\Gamma}\sqrt{1+\dot{f}(\Gamma)^2} = 0$ **(1 point)**. Computing the derivative, we obtain that:

$$\frac{\dot{f}(\Gamma)\ddot{f}(\Gamma)}{\sqrt{1+\dot{f}(\Gamma)^2}} = 0.$$

This is only the case if either $\dot{f}(\Gamma) = 0$ or if $\ddot{f}(\Gamma) = 0$, which are the conditions for $\mu(t)$ to be a geodesic. Therefore, we have proven that ν is always a geodesic. **(1 point)**

2. Prove or provide a counter example to the following statement:
For any two Riemannian metrics g, h on interval $P = (a, b) \subset \mathbb{R}$ there is a Riemannian isometry between the Riemannian charts (P, g) and (P, h).

> **Solution:** Consider the quantity $L = \int_a^b g(s)\,\mathrm{d}s$. *[Note that P is one-dimensional, so the metric $g : (a, b) \to \mathbb{R}$ is just a positive function.]* If there were an isometry $f : (a, b) \to (a, b)$, i.e. a bijective C^1-function such that $f^*h = g$, then $L = L' := \int_a^b h(s)\,\mathrm{d}s$.
>
> Indeed, since f is bijective, we can assume w.l.o.g. that $f(a) = a$, $f(b) = b$ and $f'(s) > 0$ for all s. *[The case where f is decreasing is similar.]* Moreover, by definition of the pull-back
> $$h(s) = f^*g(s) = f^*g(s)(e_1) = g(f(s))(f'(s)e_1) = g(f(s))f'(s).$$
> So, by the chain rule, $L = L'$. It's easy to find examples where $L \neq L'$. For example, we can take $g = 1$ and $g = 2$. These charts are not isometric, by the above argument.
> **(5 points)**
>
> **Remark:** Parametrization by arclength gives a *local* isometry between all one-dimensional Riemannian charts. However, the above argument shows that this cannot be a global isometry!
> **(Max: 3 points)**

3. In the Riemannian chart (P, g) defined by $P = \{(x, y) \in \mathbb{R}^2 | xy > 1\}$ and $g_{1,1}(x, y) = x$, $g_{2,2}(x, y) = y$ and $g_{1,2}(x, y) = 1$ we propose to do the following computations:
   (a) Compute the Christoffel symbol $\Gamma^2_{1,2}$.

   > **Solution:** By definition:
   > $$\Gamma^1_{12} = \frac{1}{2} g^{-1}_{11} \left(\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}\right) + \frac{1}{2} g^{-1}_{12} \left(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}\right).$$
   > It's not difficult to see that all six quantities in brackets are zero. Hence, $\Gamma^1_{12} = 0$.
   > **(4 points)**

   (b) Compute the oriented angle at $(8, 4) = \alpha(0) = \beta(0)$ between curves $\alpha, \beta : (-1, 1) \to P$ defined by $\alpha(t) = (8 - t, 4 + t)$ and $\alpha(t) = (8 + t^2, 4 - t)$. What is its measure with respect to the orientation of $\mathbb{R}^2$ containing $(-e_2, e_1)$?

   > **Solution:** The tangent vectors are given by
   > $$a := \dot{\alpha}(0) = (-1, 1), \quad b := \dot{\beta}(0) = (0, -1).$$
   > At the point $(8, 4)$, the metric is defined by
   > $$\langle v, w\rangle = v^T \begin{pmatrix} 8 & 1 \\ 1 & 4 \end{pmatrix} w.$$

So, we can calculate

$$\|a\|^2 = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}8 & 1\\ 1 & 4\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = 10$$
$$\langle a, b\rangle = \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}8 & 1\\ 1 & 4\end{pmatrix}\begin{pmatrix}0\\ -1\end{pmatrix} = -3$$
$$\|b\|^2 = \begin{pmatrix}0 & -1\end{pmatrix}\begin{pmatrix}8 & 1\\ 1 & 4\end{pmatrix}\begin{pmatrix}0\\ -1\end{pmatrix} = 4$$

(4 points)

Note that
$$\frac{\langle a, b\rangle}{\|a\|\|b\|} = \frac{-3}{2\sqrt{10}} = -\frac{3}{20}\sqrt{10},$$
which gives an associated angle of 118°. *[Note: The given orientation is equivalent to the standard one, so we do not get an extra minus-sign.]*
(2 points)