Geometry homework set 3

To be handed in 30-3 before noon.

1. Consider the surface Σ_f obtained by rotating the graph of a positive C^2 function $f : [-1,1] \to \mathbb{R}$ around the x-axes. Inspired by playing with soap films spanned between two circles we would like to find a f such that Σ_f has minimal area. Recall the area of Σ is given by the integral $2\pi \int_{-1}^{1} f(t)\sqrt{1+\dot{f}(t)^2}dt$.

   (a) Write down a differential equation that functions f minimizing the area of Σ_f should satisfy.

   (b) Verify that $f(t) = \cosh(t)$ solves the equation you found.

   (c) Setting $P = (-1,1) \times (0,2\pi)$ we get the Riemannian chart (P,g) of Σ_f using $\phi : P \to \mathbb{R}^3$ defined by $\phi(t,\theta) = (t, f(t)\cos\theta, f(t)\sin\theta)$ and with $g = \phi^* g_E$ the pull-back metric. Give a formula for the function $g_{1,2} : P \to \mathbb{R}$ in terms of f.

   (d) For constant c we consider the curves μ, λ in P given by $\mu(t) = (t,c)$ and $\lambda(t) = (c,t)$. Give an example of f such that all the μ, λ are geodesics on Σ_f.

   (e) Give an example of f such that not all curves μ are geodesics on Σ_f.

   (f) Prove that ν is always a geodesic of Σ_f where ν is obtained from μ by reparametrizing it to have constant speed (with respect to g). *Hint: prove that $\ddot{\nu}$ is orthogonal to both $\dot{\nu}$ and $\dot{\lambda}$.*

2. Prove or provide a counter example to the following statement:
   For any two Riemannian metrics g, h on interval $P = (a,b) \subset \mathbb{R}$ there is a Riemannian isometry between the Riemannian charts (P,g) and (P,h).

3. In the Riemannian chart (P,g) defined by $P = \{(x,y) \in \mathbb{R}^2 | xy > 1\}$ and $g_{1,1}(x,y) = x$, $g_{2,2}(x,y) = y$ and $g_{1,2}(x,y) = 1$ we propose to do the following computations:

   (a) Compute the Christoffel symbol $\Gamma^2_{1,2}$.

   (b) Compute the oriented angle at $(8,4) = \alpha(0) = \beta(0)$ between curves $\alpha, \beta : (-1,1) \to P$ defined by $\alpha(t) = (8-t, 4+t)$ and $\beta(t) = (8+t^2, 4-t)$. What is its measure with respect to the orientation of $\mathbb{R}^2$ containing $(-e_2, e_1)$?