Geometry Homework 2

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You can receive $3 + 5 + 8 + 4 + 4 = 24$ points. Your grade is computed $(\frac{\text{number of points}}{24}) * 9 + 1$.

1. For a Euclidean vector space E of dimension 3, express the antipodal map $a : E \to E$ defined by $a(p) = -p$ as a composition of reflections.

Solution:

Take the reflections $\sigma_{xy} : \mathbb{R}^3 \to \mathbb{R}^3$ in the hyperplane spanned by the x-axis and the y-axis. This reflection takes the point (x, y, z) to $(x, y, -z)$. Similarly define the reflections $\sigma_{yz}(x, y, z) = (-x, y, z)$ and $\sigma_{xz}(x, y, z) = (x, -y, z)$. The antipodal map can be written as the composition of the above reflections.

$$\sigma_{xy} \circ \sigma_{xz} \circ \sigma_{yz}(x, y, z) = \sigma_{xy} \circ \sigma_{xz}(-x, y, z) = \sigma_{xy}(-x, -y, z) = (-x, -y, -z).$$

(3 points)

2. Imagine a convex non-degenerate polyhedron with h hexagonal faces and p pentagonal faces. Prove that $p = 12$. (Hint: How many polygons can meet at a vertex?)

Solution: Let V, F and E be the vertices, edges and faces as usual. Let p and h denote the number of pentagons and hexagons. Moreover, let r be the number of faces meeting at a vertex. Notice that:

$$F = p + h \quad (1)$$
$$2E = 5p + 6h \quad (2)$$
$$2E = rV \quad (3)$$
$$V - E + F = 2 \quad (4)$$

(1 point)

Notice that equation (2) comes from the fact that if we count the number of edges per polygon (6 for hexagons and 5 for pentagons) and see this as the total number of edges we would count each edge exactly twice (i.e. Each edge belongs to two faces).

Additionally, $r = 3$ since the sum of the angles meeting at a point has to be strictly less than $360°$ (strictly less because a small angle is needed to make a three dimensional shape out of a plane). The angle meeting a vertex of a pentagon is 108 (total sum of angles is $(n-2) * 180°$, so $(3/5) * 180° = 108°$). The angle at a vertex of a hexagon is $(4/6) * 180 = 120°$. Only using pentagons and hexagons implies that the number of faces is at most three around each vertex.
(2 point)

Equation (3) comes from the fact that each edge has 2 ends. We note that we can rewrite this equation by writing $3V = 2V + V = 2E$ if and only if $2(-V + E) = V$ if and only if:

$$2(V - E) = -V. \tag{5}$$

(1 point)

Multiplying the Euler characteristic (4) by 6 we get the desired 12 on the right-hand side, yielding the following computation:

$$\begin{aligned} 6V - 6E + 6F = 12 &\xRightarrow{(1)\&(3)} 3 * 2(V - E) + 6(p + h) = 12 \\ &\xRightarrow{(5)} -3V + 6p + 6h = 12 \\ &\Rightarrow -3V + p + 5p + 6h = 12 \\ &\xRightarrow{(2)} p + 2E - 3V = 12 \\ &\xRightarrow{(3)} p = 12 \end{aligned}$$

Which proves the claim.
(1 point)

3. (a) In three dimensional affine Euclidean space $\mathcal{E}$ consider a regular square with vertices A, B, C, D in a plane with $\overrightarrow{AB} = \overrightarrow{DC}$ and a segment $[P, Q]$ with $\phi\overrightarrow{PQ} = \overrightarrow{AB}$. Prove that we can choose our points such that $d(A, B) = 2\phi$ and $d(P, Q) = 2 = d(P, A) = d(P, D) = d(Q, B) = d(Q, C)$. Here $\phi = \frac{1+\sqrt{5}}{2}$ is the golden number.

Solution: Draw line segment PQ above square $ABCD$ such that the projection $P'Q'$ is in the middle of the square, i.e. if N and O are the midpoints of AD and BC respectively, we choose $d(N,P') = d(O,Q') = \phi - 1$, and $d(Q',C) = d(Q',B) = d(P',A) = d(P',D)$. Moreover, choose $d(P,P') = d(Q,Q') = h$, and that $\phi\overrightarrow{PQ} = \overrightarrow{AB}$. It's clear that we can make all these choices. The goal is to tune h such that all other conditions are satisfied.
(2 points)

The triangle $QQ'O$ is right-angled, with $d(O,Q') = \phi - 1$. So, by Pythagoras,

$$d(O,Q) = \sqrt{(\phi-1)^1 + h^2} = \sqrt{\phi^2 - 2\phi + 1 + h^2} = \sqrt{2 - \phi + h^2}.$$

Also, $d(Q,B) = d(Q,C)$, so the triangle OCQ is right-angled as well. Hence, using $d(O,C) = \frac{1}{2}d(B,C) = \phi$:

$$d(Q,C) = \sqrt{(2 - \phi + h^2) + \phi^2} = \sqrt{h^2 + 3}.$$

By construction, this also equals $d(Q,B) = d(P,D) = d(P,A)$. Thus, if we choose $h = 1$, we are done.
(2 points)

(b) If K, L, M, N, O are the midpoints of segments $[A,B]$, $[C,D]$, $[P,Q]$, $[A,D]$ and $[B,C]$ respectively and the measures of the geometric angle between segments $[K,M]$ and $[K,L]$ is κ and the measure of the geometric angle between segments $[N,O]$ and $[N,P]$ is ν then prove $\kappa + \nu = \pi$.

Solution: Consider the triangles $NP'P$ and $KM'M$. Both of these are right-angled triangles, and from part (a), we know that:

$$d(N,P') = \phi - 1, \quad d(P',P) = 1, \quad d(M',M) = 1, \quad d(K,M') = \phi.$$

A boring way to deduce that $\kappa + \nu = \pi/2$ is to compute the sines, and use the addition formula. Alternatively, using that $1/\phi = \phi - 1$ shows that the two triangles are similar, and hence all their angles are equal. Hence, using the angle sum in either triangle shows that $\pi = \pi/2 + \kappa + \nu \implies \kappa + \nu = \pi/2$.
(2 points)

(c) Comment on how your answer from the previous part validates Euclid's construction of the dodecahedron.

Solution: A correct solution needs to comment on (at least) the following:

- The regular dodecahedron is constructed by adding a "roof" to each side of a regular square. (A picture suffices.)
  (1 point)

- Part (b) implies that a trapezoidal part from one roof, and a triangular part from the roof one face over are actually coplanar. Hence, together they define a regular pentagon. (Regularity follows from part (a)).
  (1 point)

4. Imagine an affine map ψ from affine 4-space $\mathcal{E}$ to itself. As usual E denotes the four-dimensional underlying vector space. Is it true that if the underlying linear map $\overrightarrow{\psi} : E \to E$ is the identity then ψ must be a translation?

Solution: Yes. Recall that an affine map $\psi : \mathcal{E} \to \mathcal{E}$ is a translation if the vector $\overrightarrow{M\psi(M)}$ is constant, i.e. independent of $M \in \mathcal{E}$.
(1 point)

Since ψ has the identity as underlying linear map, there exists a point $O \in \mathcal{E}$ such that

$$\overrightarrow{\psi(O)\psi(M)} = \overrightarrow{\psi}\left(\overrightarrow{OM}\right) = \overrightarrow{OM}$$

for all $M \in \mathcal{E}$.
(1 point)

Hence,

$$\overrightarrow{M\psi(M)} = \overrightarrow{M\psi(O)} + \overrightarrow{\psi(O)\psi(M)} = \overrightarrow{M\psi(O)} + \overrightarrow{OM} = \overrightarrow{O\psi(O)} =: u,$$

which is independent of M. Thus, ψ is a translation.
(2 points)

5. (a) Prove that any pair of distinct great circles on the 2-sphere $S^2 = \{x \in E : |x| = 1\}$ in a three-dimensional Euclidean vector space intersects in precisely two points A, B and the origin is the midpoint of the segment $[A, B]$.

Solution: The definition of a great circle on S^2 is that it is the intersection of of S^2 with a linear subspace of E of dimension 2. Let V_1 and V_2 be the two distinct subspaces that intersect S^2 to create the great circles. Notice that these planes intersect in a line l. This line passes through the origin O, as both V_1 and V_2 do (because they are subspaces). The line that passes through the origin crosses the sphere at exactly at the points A and B, which are distinct. Notice that $d(O, A) = d(O, B) = r$ proving that the origin is the midpoint of this segment $[A, B] \subset l$.
(2 points)

(b) The two distinct great circles divide up the sphere S^2 into a number of time zones. How many?

Solution: There are four distinct time zones. Notice that V_1 divides the sphere into two time zones. Similarly for V_2. Applying both V_1 and V_2 leaves us with 4 distinct time zones. (A picture would be easier than this explanation here and would considered a correct answer).
(1 point)

(c) What is the sum of the measures of the geometric angles of all the time zones from the previous part?

Solution: From part (a) we know that the pair of distinct great circles intersect precisely in the points A and B. Notice that the angles around a point have geometric angle 2π, since we have two points we know that the total geometric angle is $2\pi + 2\pi = 4\pi = 0 \mod 2\pi$. **(1 point)**