Remark about homework 6

25th October 2018

Let A be an abelian group. We can always make sense of "multiplication by integers". Namely, if $n \in \mathbb{Z}$ and $a \in A$, then we define

$$na := \begin{cases} a + \overset{n \text{ times}}{\cdots} + a, & n > 0 \\ 0, & n = 0 \\ (-a) \overset{-n \text{ times}}{\cdots} + (-a), & n < 0 \end{cases} .$$

In particular, this operation is distributive with respect the sum of integers and the sum in A, and it also satisfies $(nm)a = n(ma)$ and $1a = a$. This says that every abelian group is a $\mathbb{Z}$-module.

For $n \in \mathbb{Z}$, one can define the map "multiplication by n",

$$A \xrightarrow{\cdot n} A \qquad , \qquad a \mapsto na,$$

which is a group homomorphism.

Definition. The n**-torsion** of A is

$${}_nA := \mathrm{Ker}(A \xrightarrow{\cdot n} A) = \{a \in A : na = 0\},$$

and it is easy to see that it is a subgroup of A.

In a similar way, one can consider the image of $A \xrightarrow{\cdot n} A$,

$$nA := \mathrm{Im}(A \xrightarrow{\cdot n} A) = \{na : a \in A\},$$

which is another subgroup of A, and usually one considers the quotient A/nA.

Note that by the isomorphism theorem $A/{}_nA \simeq nA$.

Example. Let $A = \mathbb{Z}$. For any integer $n \neq 0$ the n-torsion is trivial, since $na = 0$ implies $a = 0$. Obviously, the 0-torsion is $\mathbb{Z}$. The image of "multiplication by n" is the classic subgroup $n\mathbb{Z}$, so the quotient is $\mathbb{Z}/n\mathbb{Z}$.

Example. Let $A = \mathbb{Z}/6\mathbb{Z}$. We have to distinguish some cases:

- If both 2 and 3 divide n, then 6 divides n, so $n[a] = [na] = 0$ for all $a \in \mathbb{Z}/6\mathbb{Z}$, that is, ${}_n(\mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$. Obviously $nA = 0$.
- If 2 divides n but 3 does not, then $n[a] = 0$ iff na is a multiple of 6, what happens precisely if $[a] = [0]$ or $[a] = [3]$, that is, ${}_n(\mathbb{Z}/6\mathbb{Z}) = \{[0], [3]\}$, the subgroup generated by [3]. By the isomorphism theorem, $n(\mathbb{Z}/6\mathbb{Z}) = \{[0], [2], [4]\}$, the subgroup generated by [2].
- If 2 does not divide n but 3 does, then $n[a] = 0$ iff na is a multiple of 6, what happens precisely if $[a] = [0]$, $[a] = [2]$ or $[a] = [4]$, that is, ${}_n(\mathbb{Z}/6\mathbb{Z}) = \{[0], [2], [4]\}$, the subgroup generated by [2]. By the isomorphism theorem, $n(\mathbb{Z}/6\mathbb{Z}) = \{[0], [3]\}$, the subgroup generated by [3].
- If neither 2 nor 3 divide n, then $na = 0$ implies $a = 0$ so ${}_n(\mathbb{Z}/6\mathbb{Z}) = 0$. By the isomorphism theorem, $n(\mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$.