Solutions to selected exercises

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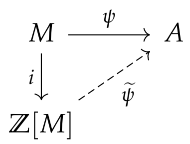
Exercise 1.2. *Let M be a set and let $i : M \longrightarrow \mathbb{Z}[M]$ be the map sending m to the function $f : M \longrightarrow \mathbb{Z}$, $f(m) = 1$ and $f(n) = 0$ for $n \neq m$. Let A be an abelian group and let $\psi : M \longrightarrow A$ be a map of sets.*

Show that there exists a unique group homomorphism $\widetilde{\psi} : \mathbb{Z}[M] \longrightarrow M$ such that $\widetilde{\psi} \circ i = \psi$.



Proof. By simplicity, recall the notation (or even better, the isomorphism $\mathbb{Z}[M] \simeq \bigoplus_M \mathbb{Z}$) we use to denote as $k_1 m_1 + \cdots + k_r m_r$ to the function $f : M \longrightarrow \mathbb{Z}$ sending $f(m_1) = k_1, \ldots, f(m_r) = k_r$ and everything else to 0 . Under this description, $i(m) = 1 \cdot m$.

Define $\widetilde{\psi}(k_1 m_1 + \cdots + k_r m_r) := k_1 \psi(m_1) + \cdots + k_r \psi(m_r)$, which is clearly a group homomorphism, since it is linear by definition. Moreover, $(\widetilde{\psi} \circ i)(m) = \widetilde{\psi}(1 \cdot m) = 1 \cdot \psi(m) = \psi(m)$.

The uniqueness follows since if $\overline{\psi}$ is another group homomorphism satisfying $\overline{\psi} \circ i = \psi$ then

$$\begin{aligned} \overline{\psi}(k_1 m_1 + \cdots + k_r m_r) &= \overline{\psi}(k_1 m_1) + \cdots + \overline{\psi}(k_r m_r) = k_1 \overline{\psi}(1 \cdot m_1) + \cdot + k_r \overline{\psi}(1 \cdot m_r) \\ &= k_1 \widetilde{\psi}(1 \cdot m_1) + \cdots + k_r \widetilde{\psi}(1 \cdot m_r) = \widetilde{\psi}(k_1 m_1 + \cdots + k_r m_r). \end{aligned}$$

□

Exercise 1.5. *If C^i is a family of chain complexes indexed by a set I, then $\bigoplus_{i \in I} C^i$ is the chain complex with $(\bigoplus_{i \in I} C^i)_n := \bigoplus_{i \in I} C^i_n$ and differential $\partial((c_i)_{i \in I}) = (\partial c_i)_{i \in I}$.*

Show that there is a canonical isomorphism

$$\Phi : \bigoplus_{i \in I} H_n(C^i) \longrightarrow H_n(\bigoplus_{i \in I} C^i).$$

Proof. Define $\Phi(([c_i])_{i \in I}) := [(c_i)_{i \in I}]$. This is a well defined group homomorphism, since $\Phi(([c_i + \partial d_i])_{i \in I}) = [(c_i + \partial d_i)_{i \in I}] = [(c_i)_{i \in I} + \partial(d_i)_{i \in I}] = [(c_i)_{i \in I}] = \Phi(([c_i])_{i \in I})$. Surjectivity is clear, and for injectivity, if $[(c_i)_{i \in I}] = 0$, then $(c_i)_{i \in I} \in \operatorname{Im} \partial$ so $(c_i)_{i \in I} = \partial(d_i)_{i \in I} = (\partial d_i)_{i \in I}$ by the definition of the differential in the direct sum. Therefore $[c_i] = [\partial d_i] = 0$ for all $i \in I$, and $([c_i])_{i \in I} = 0$. □

Exercise 2.1. *Let K be a field and let C be a chain of K-vector spaces such that only finitely many of the C_n are non-trivial vector spaces, and such that each C_n is a finite dimensional vector space.*

Show that the following equation holds:

$$\sum_{k \geq 0} (-1)^k \dim C_k = \sum_{k \geq 0} (-1)^k \dim H_k(C)$$

Proof. Since only finitely many C_n's are non-trivial, we can suppose that they are below one index $n \in \mathbb{N}$, so we have the following situation:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0$$

*These solutions are based on joined work with Guillermo Bijkerk Vila for the course *Algebraic Topology* in Autumn 2017.

Note that the equality makes sense as in both sides the sum is finite, because $C_k = 0$ and $\partial_k = 0$ (and hence $H_k(C) = 0$) for $k > n$. Since the spaces are vector spaces and the differentials are linear maps, we can use the dimension formula $\dim C_k = \dim \operatorname{Ker} \partial_k + \dim \operatorname{Im} \partial_k$, and hence

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \dim C_k &= \dim C_0 + \sum_{k=1}^{n} (-1)^k (\dim \operatorname{Ker} \partial_k + \dim \operatorname{Im} \partial_k) \\
&= \dim C_0 + \sum_{k=1}^{n} (-1)^k \dim \operatorname{Ker} \partial_k - \sum_{k=1}^{n} (-1)^{k-1} \dim \operatorname{Im} \partial_k \\
&= \dim C_0 + \sum_{k=1}^{n} (-1)^k \dim \operatorname{Ker} \partial_k - \sum_{k=0}^{n-1} (-1)^k \dim \operatorname{Im} \partial_{k+1} \\
&= \dim C_0 - \dim \operatorname{Im} \partial_1 + \sum_{k=1}^{n-1} (-1)^k (\dim \operatorname{Ker} \partial_k - \dim \operatorname{Im} \partial_{k+1}) + (-1)^n \dim \operatorname{Ker} \partial_n \\
&= \dim C_0 - \dim \operatorname{Im} \partial_1 + \sum_{k=1}^{n-1} (-1)^k (\dim \operatorname{Ker} \partial_k - \dim \operatorname{Im} \partial_{k+1}) \\
&\quad + (-1)^n (\dim \operatorname{Ker} \partial_n - \dim \operatorname{Im} \partial_{n+1}) \\
&= \sum_{k=0}^{n} (-1)^k \dim H_k(C),
\end{aligned}$$

where in the penultimate equality we have taken into account that $\operatorname{Im} \partial_{n+1} = 0$. □

Exercise 2.2. *Let $f : A \longrightarrow B$ be a abelian group homomorphism, set $Q := B / \operatorname{Im} f$ and let $q : B \longrightarrow Q$ be the canonical homomorphism to the quotient.*

(i) Show that if $p : B \longrightarrow P$ is a homomorphism of abelian groups with $p \circ f = 0$, then there exists a unique homomorphism $p' : B \longrightarrow P$ with $p' \circ q = p$.

(ii) Suppose that $q' \colon B \longrightarrow Q'$ is a homomorphism of abelian groups such that $q' \circ f = 0$ and such that for any homomorphism of abelian groups $p \colon B \longrightarrow P$ with $p \circ f = 0$, there exists a unique homomorphism $p' \colon Q' \longrightarrow P$ with $p' \circ q' = p$. Show that there exists a unique isomorphism $r \colon Q \longrightarrow Q'$ such that $r \circ q = q'$.

(iii) Let C and D be chain complexes of abelian groups and let $f \colon C \longrightarrow D$ be a chain map. For $n \geqslant 0$, let $E_n = D_n / \operatorname{Im} f_n$ and let $q_n \colon D_n \longrightarrow E_n$ be the quotient map. Show that there exists a unique family of maps $\partial_n^E \colon E_n \longrightarrow E_{n-1}$ such that

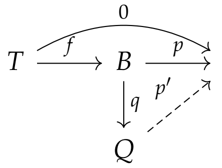
$$\cdots \xrightarrow{\partial_3^E} E_2 \xrightarrow{\partial_2^E} E_1 \xrightarrow{\partial_1^E} E_0$$

is a chain complex and the q_n define a chain map $q \colon D \longrightarrow E$.

Proof. For (i), let us define $p' : Q \longrightarrow P$ in the obvious way: $p'(\overline{b}) := p(b)$. First of all let us check that it is well-defined: another representative of $\overline{b}$ is $\overline{b + f(a)}$ for some $a \in A$, and in this case

$$p'(\overline{b + f(a)}) = p(a + f(a)) = p(a) + (p \circ f)(a) = p(a),$$

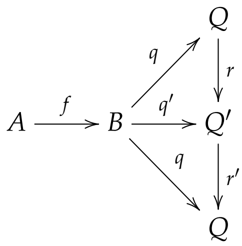
hence it does not depend on the choice of the representative of $\overline{b}$.



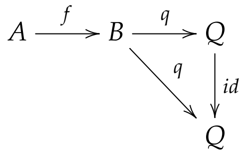
Defined in this way, it holds $p' \circ q = p$ obviously (since q is the projection to the quotient), and it is unique, because if $\varphi : Q \longrightarrow P$ was another group homomorphism satisfying $\varphi \circ q = p$, then for all $\overline{b} \in Q$ it holds

$$p'(\overline{b}) = p(b) = \varphi(q(b)) = \varphi(\overline{b}).$$

To see (ii), we consider the following diagram:



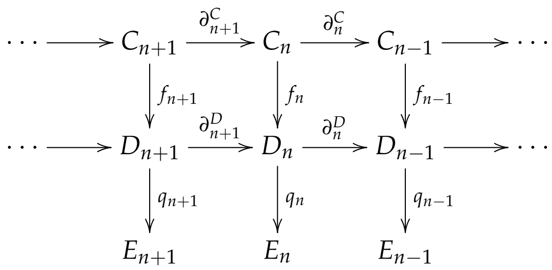
Since $q' \circ f = 0$ by assumption, following (i) we know that the morphism $r\colon Q \longrightarrow Q'$ exists and is unique, and the upper triangle commutes. Now, $q \circ f = 0$, so again by assumption $r'\colon Q' \longrightarrow Q$ also exists and is unique, and the lower triangle commutes. This means that we can just consider the big, outer triangle and take $r' \circ r\colon Q \longrightarrow Q$ as the morphism that makes it commute. But the simpler diagram



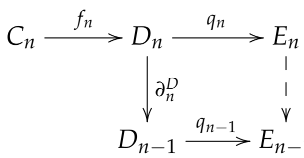
does the job as well, and so by unicity of the homomorphism $Q \longrightarrow Q$ it must be $r' \circ r = id_Q$.

By exchanging Q and Q' and the corresponding homomorphisms in the diagram above and doing an analogous reasoning, we obtain that $r \circ r' = id_{Q'}$, and therefore $r\colon Q \longrightarrow Q'$ is the desired isomorphism, having $r'\colon Q' \longrightarrow Q$ as its inverse.

For (iii), we have the following diagram:



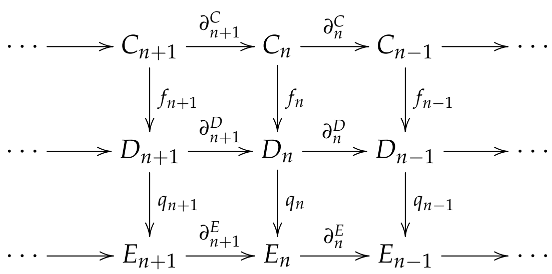
where the inner squares between C and D commute. Now consider the (sub)diagram



Note that $q_{n-1} \circ \partial^D_n$ is a group homomorphism, and

$$(q_{n-1} \circ \partial_n) \circ f_n = q_{n-1} \circ (f_{n-1} \circ \partial^C_n) = 0 \circ \partial^C_n = 0,$$

where we have used the fact that composition is associative and f is a chain map. Thus, by part (i) of this exercise, there exists a unique homomorphism $\partial^E_n\colon E_n \longrightarrow E_{n-1}$ such that the (sub)diagram commutes. This shows that the desired family of maps ∂^E_n exists and is indeed unique. Moreover, we also directly get that the q_n define a chain map $q\colon D \longrightarrow E$, because by definition the maps ∂^E_n are such that $\partial^E_n \circ q_n = q_{n-1} \circ \partial^D_n$ holds.



Now, to see that this really is a chain complex, we need to show that $\partial_n^E \circ \partial_{n+1}^E = 0$. Indeed, by commutativity of the inner squares,

$$(\partial_n^E \circ \partial_{n+1}^E)(\bar{x}) = (\partial_n^E \circ \partial_{n+1}^E \circ q_{n+1})(x) = (\partial_n^E \circ q_n \circ \partial_{n+1}^D)(x) = q_{n-1} \circ \partial_n^D \circ \partial_{n+1}^D)(x) = q_{n-1}(0) = 0,$$

where $\bar{x}$ denotes the equivalence class of x in the quotient group E_{n+1}.

(Thanks Mireia Martínez i Sellarès for this proof). □

Exercise 3.4. *(i) Let* $0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} \overline{A} \longrightarrow 0$ *be a short exact sequence of abelian groups. Show that the following statements are equivalent (such a sequence is called* ***split****):*

(a) q admits a section s.

(b) j admits a retraction r.

(c) There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{j} & A & \xrightarrow{q} & \overline{A} & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{i} & A' \oplus \overline{A} & \xrightarrow{p} & \overline{A} & \longrightarrow & 0 \end{array}$$

where f is an isomorphism and $i(a') = (a', 0)$ and $p(a', \bar{a}) = \bar{a}$.

(ii) Give an example of a short exact sequence which is not split.

(iii) Let X be a space and $X' \subset X$ a subspace, and let $r : X \longrightarrow X'$ be a retraction. Use (i) to construct an isomorphism

$$H_n(X; A) \longrightarrow H_n(X'; A) \oplus H_n(X, X'; A).$$

Proof. (i). Clearly, $(c) \implies (a)$ and $(c) \implies (b)$ by defining $r := \pi \circ f$ and $s := f^{-1} \circ \iota$, where $\pi(a', \bar{a}) = a'$ and $\iota(\bar{a}) = (0, \bar{a})$. By the commutativity of the diagram of (c), we get that $r \circ j = \mathrm{Id}_{A'}$ and $q \circ s = \mathrm{Id}_{\overline{A}}$.

$(b) \implies (c)$. Define $f := (r, q)$, that is, $f(a) := (r(a), q(a))$. It gives a commutative diagram as desired, so it remains to check that it is isomorphism: for injectivity, if $0 = f(a) = (r(a), q(a))$, the, $r(a) = 0$ and $q(a) = 0$, so $a \in \operatorname{Ker} q = \operatorname{Im} j$ by exactness, and $a = j(a')$ for some $a' \in A'$, and therefore $a' = rj(a') = r(a) = 0$, thus $a = 0$. For surjectivity, given $(a', \bar{a})$ with $\bar{a} = q(a)$ for some $a \in A$,consider $x := a' - r(a)$. Then $a + j(x)$ is the desired element, since $f(a + j(x)) = (r(a) + x, q(a) + qj(x)) = (a', \bar{a})$.

$(a) \implies (c)$. Here it is more convenient to define the isomorphism in the other direction: consider $g : A' \oplus \overline{A} \longrightarrow A$, $g(a', \bar{a}) := j(a') + s(\bar{a})$. For injectivity, if $g(a', \bar{a}) = 0$, then $s(\bar{a}) = -j(a') \in \operatorname{Im} j = \operatorname{Ker} q$, so $0 = qs(\bar{a}) = \bar{a}$, and also $a' = 0$ as $j(a') = 0$ and j is injective. For surjectivity, given $a \in A$, set $\bar{a} = q(a)$. We see that $a - s(\bar{a}) \in \operatorname{Ker} q = \operatorname{Im} j$, so $a - s(\bar{a}) = j(a')$ for some $a' \in A'$ and we are in business.

(ii). Consider the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$, with $n \geq 2$. Then q cannot have a section s: indeed, in such a case, if $s([1]) = k$, then $0 = s([n \cdot 1]) = nk$, so $k = 0$ and $s = 0$, what cannot happen if s is a section.

Another more direct way to argue is: $\mathbb{Z}$ is free torsion, whereas $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is not.

(iii). For every $n \in \mathbb{N}$, the retraction r induces a retraction $r_* : C_n(X; A) \longrightarrow C_n(X'; A)$ by functioriality, which is a chain map. By (i), there are isomorphisms $(r_*, q) : C_n(X; A) \longrightarrow C_n(X'; A) \oplus C_n(X, X'; A)$, where $q : C_n(X; A) \longrightarrow C_n(X, X'; A)$. Being r_* and q chain maps, so is

$$(r_*, q) : C(X; A) \longrightarrow C(X'; A) \oplus C(X, X'; A).$$

We conclude by applying exercise 1.5. □

Exercise 4.2. *Let X be a topological space and let $X'' \subset X' \subset X$ be subspaces, and let A be an abelian group. Let*

$$H_n(X', X''; A) \longrightarrow H_n(X, X''; A)$$

be the map induced by the inclusion $X' \longrightarrow X$, let

$$H_n(X, X''; A) \longrightarrow H_n(X, X'; A)$$

be the map induced by the identity of X and let

$$H_n(X, X'; A) \longrightarrow H_{n-1}(X', X''; A)$$

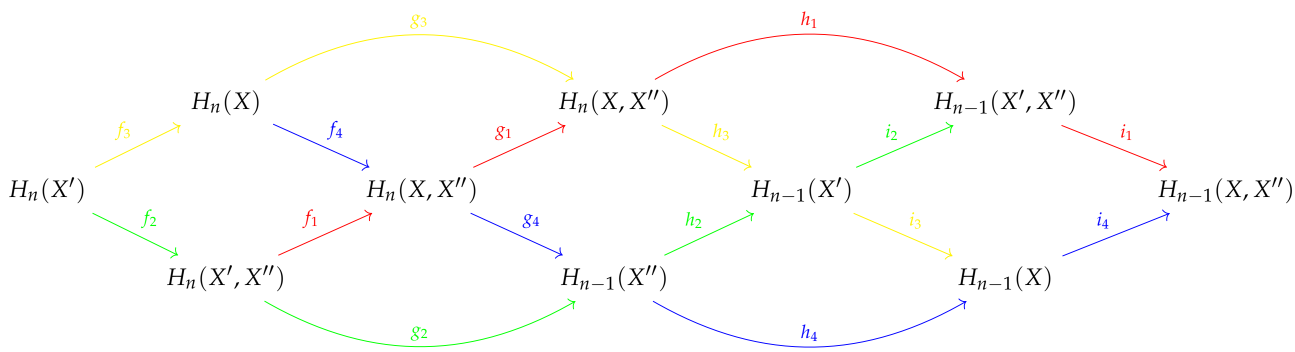
be the composite of the connecting homomorphism $H_n(X, X'; A) \longrightarrow H_{n-1}(X'; A)$ from the long exact sequence of the pair (X, X') and the map $H_{n-1}(X'; A) \longrightarrow H_{n-1}(X', X''; A)$ from the long exact sequence of the pair (X', X'').

Show that the sequence

$$\cdots \longrightarrow H_n(X', X''; A) \longrightarrow H_n(X, X''; A) \longrightarrow H_n(X, X'; A) \xrightarrow{\partial_n} H_{n-1}(X', X''; A) \longrightarrow H_{n-1}(X, X''; A) \longrightarrow H_{n-1}(X, X'; A) \xrightarrow{\partial_{n-1}} H_{n-2}(X', X''; A) \longrightarrow \cdots$$

is exact.

Proof. The first step is to make sure that the long exact sequences of the pairs (X', X''), (X, X'') and (X, X') can be interlaced to form the following "braid diagram" (I will omit the coefficient from the notation):



The braid commutes by the naturality of the long exact sequence of a pair (induced by the maps of pairs $(X', X'') \longrightarrow (X, X'')$ and $(X, X'') \longrightarrow (X, X')$). Now the claim is that if the green, yellow and blue sequences are exact, so is the red one, which is the desired one (note that h_1 here is precisely defined as $i_2 \circ h_3$, as required). The exercise here boils down to a diagram chasing problem of abelian groups. It needs to be check all possible inclusions one by one. I will just make one of them explicit:

Let's see for example that $\operatorname{Ker} g_1 \subset \operatorname{Im} f_1$: if $a \in H_n(X, X'')$, as $0 = h_3 g_1 a = h_2 g_4 a$,there is $b \in H_n(X', X'')$ such that $g_2 b = g_4 a$. Note that $g_4(f_1 b - a) = 0$, so $f_1 b - a \in \operatorname{Ker} g_4 = \operatorname{Im} f_4$, so there is $d \in H_n(X)$ such that $f_4 d = f_1 b - a$. As $g_3 = g_1 \circ f_4$, $d \in \operatorname{Ker} g_3 = f_3$, so there is $e \in H_n(x')$ such that $f_3 e = d$. Finally, $b - f_2 e$ is the desired element: indeed,

$$f_1(b - f_2 e) = f_1 b - f_1 f_2 e = f_1 b - f_4 f_3 e = f_1 b - f_4 d = f_1 b - f_1 b + a = a.$$

□

Alternative proof. Another approach is to come up with a short exact sequence of chain complexes inducing the previous long exact sequence in homology. Indeed,

$$0 \longrightarrow C(X', X''; A) \longrightarrow C(X, X''; A) \longrightarrow C(X, X'; A) \longrightarrow 0$$

is the desired sequence (the homomorphisms are induced by the inclusion $C(X'; A) \longrightarrow C(X; A)$ and the identity of $C(X; A)$, respectively). It remains to the reader to check that it is exact, that the arrows are chain maps, and that the induced morphisms in homology are the one described before. For the homomorphism corresponding to the composite, recall that elements of $H_n(X, X'; A)$ are represented by chains $c \in C_n(X; A)$ such that $\partial_n c \in C_{n-1}(X'; A)$. □

Exercise 5.1. *Let X be a topological space and A an abelian group. Let $U, V \subset X$ satisfying $X = \overset{\circ}{U} \cup \overset{\circ}{V}$, and let*

$$\begin{array}{ccccc} & & U & & \\ & \overset{i^U}{\nearrow} & & \overset{j^U}{\searrow} & \\ U \cap V & & & & X \\ & \underset{i^V}{\searrow} & & \underset{j^V}{\nearrow} & \\ & & V & & \end{array}$$

be the natural inclusion maps. Define groups homomorphisms

$$\partial_n : H_n(X; A) \longrightarrow H_{n-1}(U \cap V; A)$$

such that the sequence of groups homomorphisms

$$\cdots \longrightarrow H_n(U \cap V; A) \xrightarrow{(i_*^U, i_*^V)} H_n(U; A) \oplus H_n(V; A) \xrightarrow{\begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}} H_n(X; A) \xrightarrow{\partial_n}$$

$$H_{n-1}(U \cap V; A) \longrightarrow H_{n-1}(U; A) \oplus H_{n-1}(V; A) \longrightarrow H_{n-1}(X; A) \xrightarrow{\partial_{n-1}}$$

$$H_{n-2}(U \cap V; A) \longrightarrow \cdots$$

is exact. This is called the ***Mayer-Vietoris sequence*** *of the cover $\{U, V\}$*

Proof. Consider the sequence of chain maps that levelwise is

$$0 \longrightarrow C_n(U \cap V; A) \xrightarrow{(i_*^U, i_*^V)} C_n(U; A) \oplus C_n(V; A) \xrightarrow{\begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}} C_n(\mathcal{S}_{\mathcal{O}}; A) \longrightarrow 0 \qquad (*)$$

and note that it is enough to check that it is exact, because since by hypothesis the cover $\mathcal{O} := \{U, V\}$ is admissible, the inclusion $\mathcal{S}_{\mathcal{O}}(X)$ in $\mathcal{S}(X)$ induces an isomorphism (small simplices theorem)

$$H_n(C(\mathcal{S}_{\mathcal{O}}(X); A)) \xrightarrow{\sim} H_n(X; A),$$

and therefore the long exact sequence of the previous sequence $(*)$ will give the requested diagram (and the morphisms ∂_n are precisely the composite of the connecting homomorphisms coming from the corresponding long exact sequence with the previous isomorphism coming from the small simplices theorem). Note that here we must also take into account that $H_n(C \oplus D; A) = H_n(C; A) \oplus H_n(D; A)$ for chain complexes C, D (exercise 5, sheet 1).

For the sake of simplicity, call $i = (i_*^U, i_*^V)$ and $j = \begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}$ in $(*)$, and let us check its exactness:

- i injective:

  Note that i injective iff i_*^U injective and i_*^V injective, but both are because they precisely are the induced morphisms by the inclusions.

- j surjective:

  If $a\sigma \in C_n(\mathcal{S}_{\mathcal{O}}(X); A)$, then $\sigma \in \mathcal{S}(U)_n$ or $\sigma \in \mathcal{S}(V)_n$; so $a\sigma$ is the image of either $a\sigma$ or $-a\sigma$, respectively. For a finite sum of elements of $C_n(\mathcal{S}_{\mathcal{O}}(X); A)$ we just consider the corresponding sum of elements in the direct sum.

- $\operatorname{Im} i = \operatorname{Ker} j$:

  If $a\sigma \in C_n(U \cap V; A)$, i sends it to $(a\sigma, a\sigma) \in \operatorname{Im} i$, and by j it goes to 0, and linearity proves it for a sum. Conversely, if an element $(a\sigma, b\sigma')$ is in the kernel of j, then $a\sigma - b\sigma = 0 \in C_n(\mathcal{S}_{\mathcal{O}}(X); A)$. This implies that $\sigma = \sigma'$ and $a = b$; hence $\sigma \in \mathcal{S}(U \cap V)_n$.

$\square$

Remark. The Mayer-Vietoris sequence is also natural, that is, if $f : X \longrightarrow Y$ is a map of spaces and $X = U_1 \cup V_1$ and $Y = U_2 \cup V_2$ with $f(U_1) \subset U_2$ and $f(V_1) \subset V_2$, then there is a commutative diagram relating both sequences, as for the long exact sequence of a pair. This follows from the fact that the Mayer-Vietoris sequence comes up from the long exact sequence induced by a certain short exact sequence of chain maps. The conditions on the covers implies that there is a commutative diagram of chain maps with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(U_1 \cap V_1; A) & \longrightarrow & C(U_1; A) \oplus C(V_1; A) & \longrightarrow & C(\mathcal{S}_{\mathcal{O}}(X); A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(U_2 \cap V_2; A) & \longrightarrow & C(U_2; A) \oplus C(V_2; A) & \longrightarrow & C(\mathcal{S}_{\mathcal{O}}(Y); A) & \longrightarrow & 0 \end{array}$$

which induces the desired commutative diagram relating the Mayer-Vietoris sequences.

Remark. The Mayer-Vietoris sequence also holds for de Rham cohomology: for a smooth manifold M of dimension n, the differential of forms d induces a *cochain complex*

$$0 \longrightarrow C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

(the "*co*" means that the degree goes up), because $d^2 = 0$, and one can take *co*homology groups $H^k_{dR}(M) := \operatorname{Ker} d / \operatorname{Im} d$. For two open sets $U, V \subset M$ such that $U \cup V = M$, the inclusions

$$U \cap V \rightrightarrows U \coprod V \longrightarrow M$$

induce exact sequences

$$0 \longrightarrow \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V) \longrightarrow 0$$

which induce the Mayer-Vietoris sequence in de Rham cohomology.

Exercise 5.2. *Compute the homology groups $H_n(\mathbb{S}^m; A)$ using the Mayer-Vietoris sequence.*

Proof. This is a very easy exercise and I will not spell it out. The proof goes by induction: for $m = 0$ it boils down to the computation of the homology groups of the one-point space. For $m > 0$, one take the open sets

$$U = \{(x_0, \ldots, x_m) \in \mathbb{S}^m : x_m > -1/3\}$$

and

$$V := \{(x_0, \ldots, x_m) \in \mathbb{S}^m : x_m < 1/3\},$$

which are contractible; and one has that $U \cap V$ is homotopy equivalent to $\mathbb{S}^{m-1}$. $\square$

Exercise 5.3. *Let X be a topological space, A an abelian group and $n \geq 0$ an integer. Let $X' \subseteq X$ be a non-empty closed subspace of X that is a neighbourhood deformation retract. As usual, we write X/X' for the quotient of X that identifies all points in X'.*

(i) Show that the quotient map $X \longrightarrow X/X'$ induces an isomorphism of relative homology groups

$$H_n(X, X'; A) \xrightarrow{\sim} H_n(X/X', X'/X'; A).$$

(ii) Show that the relative homology groups $H_n(X, X'; A)$ are isomorphic to the homology groups $H_n(X/X'; A)$, if $n \geq 1$, and to $H_0(X/X'; A)/A$ if $n = 0$.

Proof. (i). Let us consider the following commutative diagram, where U is the neighbourhood of X' such that X' is a deformation retract of U (and therefore X' is homotopy equivalent to U):

$$\begin{array}{ccccc} H_n(X, X'; A) & \xrightarrow{\quad f \quad} & H_n(X, U; A) & \xleftarrow{\quad g \quad} & H_n(X - X', U - X'; A) \\ \downarrow{\pi_*} & & \downarrow & & \downarrow{\widetilde{\pi_*}} \\ H_n(X/X', X'/X'; A) & \xrightarrow{h} & H_n(X/X', U/X'; A) & \xleftarrow{i} & H_n(X/X' - X'/X', U/X' - X'/X'; A) \end{array}$$

We will show that π_* is an isomorphism by showing that the rest of labelled arrows in the diagrams are too. By the "2 out of 3" property in the following diagram, we see that f is an isomorphism,

$$\begin{array}{ccccc} H_n(X';A) & \longrightarrow & H_n(X;A) & \longrightarrow & H_n(X,X';A) \\ \downarrow \wr & & \downarrow \wr & & \downarrow f \\ H_n(U;A) & \longrightarrow & H_n(X;A) & \longrightarrow & H_n(X,U;A) \end{array}$$

because the inclusion $X' \longrightarrow U$ is a homotopy equivalence. Moreover, the homotopy $H : U \times I \longrightarrow U$ defines another homotopy $\bar{H} : U/X' \times I \longrightarrow U/X'$, continuous and well defined by the universal property of the quotient topology,

$$\begin{array}{ccccc} U \times I & \xrightarrow{H} & U & \longrightarrow & U/X' \\ \downarrow & & & \overset{\bar{H}}{\nearrow} & \\ U/X' \times I & & & & \end{array}$$

since for $x_1', x_2' \in X'$, we have $H(x_1',t) \in X'$, $H(x_2',t) \in X'$, and hence they are the same point in the quotient U/X'. In particular, the previous diagram says that X'/X' is a deformation retract of U/X' with homotopy $\bar{H}$. One concludes that h is isomorphism.

But we also have, by excision theorem, that g and i are also isomorphisms. For that just note that one can apply the theorem since $\overline{X'} = X' \subset O = \mathring{O} \subseteq \mathring{U}$, where O is the open set such that $X' \subseteq O \subseteq U$; and the same analogously with the quotient space (because the definition of the quotient topology).

Finally, it turns out that $\widetilde{\pi_*}$ is also isomorphism trivially, since

$$\widetilde{\pi}_{|X-X'} : X - X' \longrightarrow (X - X')/X' = X/X' - X'/X',$$

$$\widetilde{\pi}_{|U-X'} : U - X' \longrightarrow (U - X')/X' = U/X' - X'/X',$$

are homeomorphisms (there's nothing to identify!).

Altogether, one has that π_* is a group isomorphism because it is composite of isomorphisms.

(ii). Just note that the previous isomorphism tells us that

$$H_n(X,X';A) = H_n(X/X',X'/X';A) = H_n(X/X',*;A) = \begin{cases} H_n(X/X';A) & n > 0 \\ H_0(X/X';A)/H_0(*;A) & n = 0 \end{cases},$$

where in the last equality we have used the long exact sequence of the pair $(X/X',*)$. $\square$

Exercise 6.3. *Let X be a topological space, A an abelian group and $n \geq 0$.*

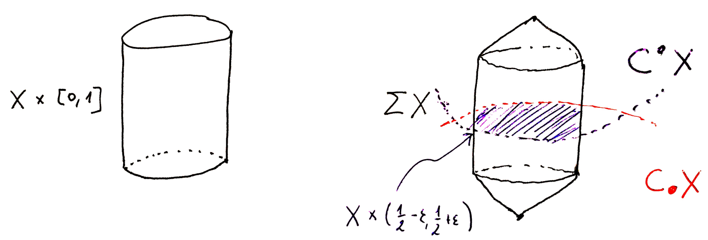
(i) Construct a natural isomorphism $\widetilde{H}_n(X;A) \simeq \widetilde{H}_{n+1}(\Sigma X;A)$ relating the n-th reduced homology group of X to the $(n+1)$-st reduced homology group of its suspension.

(ii) What is ΣS^n?

Proof. (i). Consider the cones

$$C_\bullet X := \frac{X \times [0, 1/2 + \varepsilon]}{X \times \{0\}} \quad , \quad C^\bullet X := \frac{X \times [1/2 - \varepsilon, 1]}{X \times \{1\}},$$

and note that they form a cover of the suspension ΣX.



Analogously to the unreduced case, there are Mayer-Vietoris sequences and long exact sequences in reduced homology. By applying the reduced Mayer-Vietoris sequence to this cover, we obtain that the following sequence is exact:

$$\widetilde{H}_{n+1}(X\times(1/2-\varepsilon,1/2+\varepsilon);A)\longrightarrow \widetilde{H}_{n+1}(C_\bullet X;A)\oplus H_{n+1}(C^\bullet X;A)\longrightarrow \widetilde{H}_{n+1}(\Sigma X;A)$$
$$\hookrightarrow \widetilde{H}_n(X\times(1/2-\varepsilon,1/2+\varepsilon)\longrightarrow \widetilde{H}_n(C_\bullet X;A)\oplus H_n(C^\bullet X;A)\longrightarrow \widetilde{H}_n(\Sigma X;A)$$
$$\hookrightarrow \widetilde{H}_{n-1}(X\times(1/2-\varepsilon,1/2+\varepsilon);A)\longrightarrow\cdots$$

Now, $X\times(1/2-\varepsilon,1/2+\varepsilon)$ is homotopy equivalent to X via a deformation retraction, and both $C_\bullet X$ and $C^\bullet X$ are contractible[1], hence $\widetilde{H}_n(C_\bullet X;A)\oplus\widetilde{H}_n(C^\bullet X;A)=0$ for all $n\geq 0$, so the previous exact sequence becomes

$$\cdots\longrightarrow\widetilde{H}_{n+1}(X;A)\longrightarrow 0\longrightarrow\widetilde{H}_{n+1}(\Sigma X;A)\longrightarrow\widetilde{H}_n(X;A)\longrightarrow 0\longrightarrow\cdots,$$

and therefore we obtain an isomorphism (given by the connecting homomorphism)

$$\widetilde{H}_{n+1}(\Sigma X;A)\simeq\widetilde{H}_n(X;A).$$

The statement about naturality means that if Y is another topological space and $f:X\longrightarrow Y$ is a continuous map, then we get a commutative diagram

$$\begin{array}{ccc}\widetilde{H}_{n+1}(\Sigma X;A) & \xrightarrow{\sim} & \widetilde{H}_n(X;A)\\ {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle f_*}\\ \widetilde{H}_{n+1}(\Sigma Y;A) & \xrightarrow{\sim} & \widetilde{H}_n(Y;A)\end{array}$$

where F_* is induced by the map $F:\Sigma X\longrightarrow\Sigma Y$, $F([(x,t)])=[(f(x),t)]$ induced by f in the suspension of the spaces. This follows from the naturality of the Mayer-Vietoris sequence for the cover $C_\bullet Y\cup C^\bullet Y=\Sigma Y$ and the corresponding induced morphisms (see remark after 5.1).

(ii). We will prove that $\Sigma\mathbb{S}^n=\mathbb{S}^{n+1}$. For the sake of simplicity, we will take the suspension of a topological space X as the quotient $(X\times[-1,1])/\sim$ where we identify the points in $X\times\{-1\}$ and in $X\times\{1\}$.

Consider the continuous map

$$\begin{array}{ccc}\mathbb{S}^n\times[-1,1] & \longrightarrow & \mathbb{S}^{n+1}\\ \Big((x_0,\ldots,x_n),t\Big) & \longmapsto & (\sqrt{1-t^2}x_0,\ldots,\sqrt{1-t^2}x_n,t)\end{array}$$

It satisfies that equivalent points via $\sim$ are mapped to the same point (all the points in $\mathbb{S}^n\times\{-1\}$ and $\mathbb{S}^n\times\{1\}$, that are equivalent, are mapped onto the south and north pole, respectively). Therefore, by the Universal Property of the quotient topology, we get a continuous map

$$\varphi:\frac{\mathbb{S}^n\times[-1,1]}{\sim}=\Sigma\mathbb{S}^n\longrightarrow\mathbb{S}^{n+1}.$$

We see that it is a bijection, as its inverse morphism is

$$\begin{array}{ccc}\frac{\mathbb{S}^n\times[-1,1]}{\sim}=\Sigma\mathbb{S}^n & \longleftarrow & \mathbb{S}^{n+1}\\ \left[\Big((\frac{y_0}{\sqrt{1-y_{n+1}^2}},\ldots,\frac{y_n}{\sqrt{1-y_{n+1}^2}}),y_{n+1}\Big)\right] & \longleftarrow & (y_0,\ldots,y_n,y_{n+1})\end{array}$$

[1]A cone is always contractible: in general, given a topological space X, define $CX:=\frac{X\times[0,1]}{X\times\{0\}}$. It turns out that the map $\hat{H}:CX\times[0,1]\longrightarrow CX$, $\hat{H}([(x,t)],s)=[(x,(1-s)t)]$ is a homotopy in CX from the identity on CX to the point $X\times\{0\}$, hence CX is contractible.

Since the source is a compact space, and the target is a Hausdorff space, we conclude that φ is homeomorphism, as required. $\square$

Exercise 7.2. *Let X be a topological space and let $f, g : \partial\mathbb{D}^n \longrightarrow X$ be two continuous maps. Let $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ be the space obtained by attaching an $n-$cell to X with attaching map f and let $X_g \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ be the space obtained by attaching an n-cell with attaching map g.*

Show that $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ and $X_g \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ are homotopy equivalent if f and g are homotopic.

Proof. Let $H : \partial\mathbb{D}^n \times I \longrightarrow X$ be the homotopy between f and g. The claim is that $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ is homotopy equivalent to $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$, and similarly $X_g \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ is homotopy equivalent to $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$. As the homotopy equivalence is an equivalence relation, we are done with the claim.

We will show that $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ is homotopy equivalent to $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$ (with g is analogous). The key property for that is that the cylinder $\mathbb{D}^n \times I$ deformation retracts to $\partial D^n \times I \cup \mathbb{D}^n \times I$, as we used in proposition 11.10 in the lecture notes. Denote as $F : (\mathbb{D}^n \times I) \times I \longrightarrow \mathbb{D}^n \times I$ the homotopy relating $F_0 = \mathrm{Id}_{\mathbb{D}^n \times I}$ with the retraction $F_1 = r$.

We will now prove that $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$ deformation retracts to $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n = X_{H_0} \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times \{0\} \subset X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$. By lemma 7.12,

$$(X_H \cup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I) \times I \cong X \times I_{H \times \mathrm{Id}} \cup_{(\partial\mathbb{D}^n \times I) \times I} (\mathbb{D}^n \times I) \times I,$$

so to give a homotopy from $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$ to another space is the same thing as to give homotopies from $X \times I$ and $\mathbb{D}^n \times I$ to the space which "coincide" in $\partial\mathbb{D}^n \times I$.

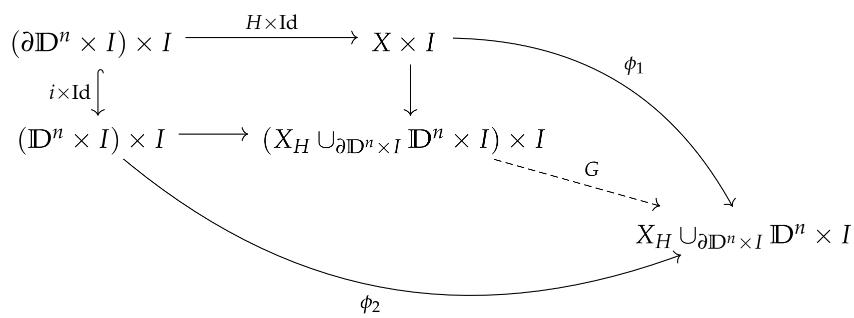
Denote as π_1 and π_2 the projections from X and $\mathbb{D}^n \times I$ to the pushout $X_H \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$, respectively, and define ϕ_1 as the composite

$$X \times I \xrightarrow{\mathrm{pr}_1} X \xrightarrow{\pi_1} X_H \cup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$$

and ϕ_2 as the composite

$$(\mathbb{D}^n \times I) \times I \xrightarrow{F \times \mathrm{Id}} \mathbb{D}^n \times I \xrightarrow{\pi_2} X_H \cup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I,$$

so we get the diagram of below with solid arrows (and the square is a pushout diagram):



Moreover, for $((y,t),s) \in (\partial\mathbb{D}^n \times I) \times I$,

$$\begin{aligned}(\phi_1 \circ (H \times \mathrm{Id}))((y,t),s) &= \phi_1(H(y,t),s) = [H(y,t)] = [(y,t)] = \phi_2((y,t),s) \\ &= (\phi_2 \circ (i \times \mathrm{Id}))((y,t),s)\end{aligned}$$

as $F_{|\partial\mathbb{D}^n \times I} = \mathrm{Id}$. Therefore, the outer diagram commutes and by the universal property of the pushout we get the dashed arrow G. This is the desired homotopy which deformation retracts $X_H \cup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times I$ into $X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$. Indeed, $G_0 = \mathrm{Id}$ and G_1 maps to $X_H \bigcup_{\partial\mathbb{D}^n \times I} (\partial D^n \times I \cup \mathbb{D}^n \times I) \cong X_{H_0} \bigcup_{\partial\mathbb{D}^n \times I} \mathbb{D}^n \times \{0\} = X_f \bigcup_{\partial\mathbb{D}^n} \mathbb{D}^n$ since all points in $\partial\mathbb{D}^n \times I$ are already identified with points of X in the pushout. $\square$

Exercise 9.2. *Construct a CW structure on $\mathbb{S}^\infty$ and show that it is contractible.*

Proof. Consider the flitration

$$\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \cdots \hookrightarrow \mathbb{S}^\infty,$$

where $\mathbb{S}^n$ is the n-skeleton and as we saw $\mathbb{S}^{n+1}$ arises from $\mathbb{S}^n$ by attaching two $(n+1)$-cells. Observe that by definition, $\mathbb{S}^\infty = \bigcup_{n\geq 0} \mathbb{S}^n$ is topologized with the subspace topology of $\mathbb{R}^\infty$, so it remains to check that the weak topology in $\mathbb{S}^\infty$ induced by the filtration is the same as the subspace topology of $\mathbb{R}^\infty$. This can be seen, for instance, using the commutativity of the diagram

$$\begin{array}{ccccccccc} \cdots & \hookrightarrow & \mathbb{S}^n & \hookrightarrow & \mathbb{S}^{n+1} & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{S}^\infty \\ & & \downarrow & & \downarrow & & & & \downarrow \\ \cdots & \hookrightarrow & \mathbb{R}^n & \hookrightarrow & \mathbb{R}^{n+1} & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{R}^\infty \end{array}$$

To see that $\mathbb{S}^\infty$ is contractible, we use the map $f : \mathbb{S}^\infty \longrightarrow \mathbb{S}^\infty$, $f(x_0, x_1, x_2, \ldots) := (0, x_0, x_1, \ldots)$. The claim is that f is homotopic both to the identity and to const_p, $p = (1, 0, 0, \ldots)$, so $\mathbb{S}^\infty$ is contractible. Indeed, we use the homotopies

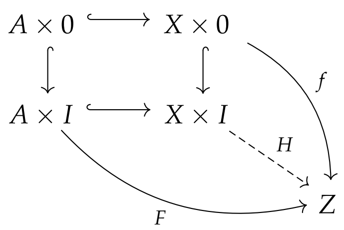
$$H : \mathbb{S}^\infty \times I \longrightarrow \mathbb{S}^\infty \qquad , \qquad H(x,t) := \frac{(1-t)x + tf(x)}{|(1-t)x + tf(x)|}$$

and

$$F : \mathbb{S}^\infty \times I \longrightarrow \mathbb{S}^\infty \qquad , \qquad F(x,t) := \frac{(1-t)f(x) + t\,\text{const}_p(x)}{|(1-t)f(x) + t\,\text{const}_p(x)|}.$$

It is readily verified that in both cases the denominator does not vanish, so the maps are continuous. □

Remark. A good way to remember the definition of the Homotopy Extension Property (HEP) of a pair (X, A) is that given the below commutative diagram with solid arrows, there exits a dashed arrow such that both triangles commute,



Exercise 11.1. *Let (X, A) be a pair of spaces with the homology extension property and assume that the inclusion map $i : A \longrightarrow X$ is homotopic to a constant map. Let $p : X \longrightarrow X/A$ be the quotient map to the space obtained from X by collapsing A to a point. Show that there exists a continuous map $r : X/A \longrightarrow X$ such that $r \circ p$ is homotopic to the identity on X*

Proof. Let $F : A \times [0,1] \longrightarrow X$ be the homotopy between the inclusion $i = F_0$ and the constant map $F_1 = \text{const}_{x_0}$, and set $f = \text{Id}_X$ in the diagram of the remark. By the HEP of the pair (X, A), F extends to a homotpy $H : X \times I \longrightarrow X$. Consider $H_1 : X \longrightarrow X$. By the universal property of the quotient topology, there exists a unique continuous map $r : X/A \longrightarrow X$ such that $H_1 = r \circ p$, since $H_1(a) = x_0$ for all $a \in A$. We conclude as $H_1 \simeq H_0 = \text{Id}_X$. □

Exercise 11.2. *Let (X, A) be a pair of spaces with the homotopy extension property and assume that A is a contractible space.*

(i) Show that the quotient map $p : X \longrightarrow X/A$ is a homotopy equivalence.

(ii) Give an example of a pair of spaces that satisfy the assumptions of Exercise 11.1 but not the assumptions of the present exercise.

Proof. (i). Being A contractible, the identity on such space is homotopic to a constant map. Composing the inclusion i with such a homotopy yields a homotopy $F : A \times [0,1] \longrightarrow X$ for which $i \simeq \text{const}_{x_0}$. Then the previous exercise provides a homotopy H and a map r such that $r \circ p \simeq \text{Id}_X$, so it remains to show that $p \circ r \simeq \text{Id}_{X/A}$.

The homotopy H induces a homotopy $\widetilde{H} : X/A \times [0,1] \longrightarrow X/A$ arising in the following form: consider the composite $p \circ H : X \times I \longrightarrow X/A$ and the map $p \times \mathrm{Id} : X \times I \longrightarrow X/A \times I$, which is a quotient map as I is locally compact (proposition 7.15). For a point $(a,t) \in X \times I$ with $a \in A$, $(p \circ H)(a,t) = [a] \in X/A$ since $H_{|A\times I} = F$ and F has image in A. Therefore, by the universal property of the quotient topology, there exists $\widetilde{H} : X/A \times [0,1] \longrightarrow X/A$. This is the desired homotopy between $\mathrm{Id}_{X/A}$ and $p \circ r$, as

$$\widetilde{H}([x],0) = [H(x,0)] = [x]$$

and

$$\widetilde{H}([x],1) = [H(x,1)] = p(H(x,1)) = (p \circ r \circ p)(x) = (p \circ r)[x].$$

(ii). Take $(\mathbb{D}^2, \partial\mathbb{D}^2 = \mathbb{S}^1)$. This pair has the HEP (proposition 11.10), and the inclusion $i : \mathbb{S}^1 \longrightarrow \mathbb{D}^2$ is homotopic to a constant map ($\pi_1(\mathbb{D}^2) = 0$), but $\mathbb{S}^1$ is not contractible as space □