Solutions Exam algebraic topology 1, 1-23-2019

Always motivate your answers and state the theorems/results you are using. Unless stated otherwise all homology is taken with integer coefficients.

Question 1

a. For a pair of spaces (X, Y) define $Z = ((Y \times [0,1]) \sqcup X)/\sim$ where $(y,1) \sim y$ and $(y,0) \sim (y',0)$ for all $y, y' \in Y$. Show that for all $n \in \mathbb{N}$ we have $H_n(X,Y) \cong \widetilde{H}_n(Z)$.

b. Assume Y from part a. is closed in X. Prove or give a counter example: if the pair (X, Y) has the homotopy extension property then so does the pair (Z, Y).

c. Explicitly describe a 2-chain representing an element of $H_2(\mathbb{R}^3, \mathbb{R}^2 - \{0\})$ with the property that its image under the connecting homomorphism for the pair $(\mathbb{R}^3, \mathbb{R}^2 - \{0\})$ is non-zero. Here $\mathbb{R}^2$ refers to the subspace spanned by the first two coordinates.

Part a. we have $\widetilde{H}_n(Z) \cong H_n(Z, CY) \cong H_n(Z - \{v\}, CY - \{v\}) \cong H_n(X, Y)$, where $v = [(y,0)]$ is the tip of the cone $CY = Y \times [0,1]/(Y \times \{0\})$. We used excision in the second isomorphism.
Part b. Y is still a closed subspace of Z so by Corollary 11.9 in chapter 11 it has HEP iff there is a retraction of $Z \times [0,1]$ onto $Z \times \{0\} \cup Y \times [0,1]$. Since there is a retraction of Z retracts onto X that restricts to the identity on Y and the pair (X, Y) has HEP, so does the pair (Z, Y).
Part c. Take a 2-simplex $\sigma : \Delta^2 \to \mathbb{R}^3$ such that $\sigma(ae_1 + be_2 + ce_3) = (1-c)(\cos \frac{2\pi a}{a+b})e_1 + (1-c)(\sin \frac{2\pi a}{a+b})e_2 + ce_3$. We see that $\partial\sigma$ is the unit circle in $\mathbb{R}^2$ which represents a generator of $H_1(\mathbb{R}^2 - \{0\})$.

Question 2

a. For sets S, T, U and maps $f : S \to T$, $g : U \to T$ define the product

$$S \times_T U = \{(s,u) \in S \times U | f(s) = g(u)\}$$

Define $F(X, Y)$ to be the set of continuous maps between spaces X, Y. Prove that $F(X \cup_A B, Y)$ is in bijection with $F(X,Y) \times_{F(A,Y)} F(B,Y)$ for a suitable choice of maps to $F(A,Y)$.

b. Recall that $\mathbb{RP}^2$ arises from $\mathbb{RP}^1$ by attaching one 2-cell. Using this description, show that $\mathbb{RP}^2$ minus a point, deformation retracts to $\mathbb{RP}^1$ by giving an explicit homotopy.

Part a. Take the maps $f : F(X,Y) \to F(A,Y)$ that precomposes with the map $A \to X$ implicit in the definition of $X \cup_A B$. Likewise for $g : F(B,Y) \to F(A,Y)$. Suppose we have a continuous map $h : X \cup_A B \to Y$ then precomposing with the quotient map yields a map from $X \sqcup B$ to Y and hence an element of $F(X,Y) \times F(B,Y)$ whose two components agree when precomposed with the above maps from A. In the other direction we use the universal property of the pushout $X \cup_A B \to Y$ to produce a continuous h as above from continuous maps into Y defined on X and B that are compatible in the sense of the product. part b. D^2 minus a point, say 0, deformation retracts onto its boundary by $H(x,t) = (1-t)x + t\frac{x}{|x|}$. $\mathbb{RP}^2$ minus a point is of the form $\mathbb{RP}^1 \cup_A B$ where B is the D^2 minus a point and A is the 1-cell in $\mathbb{RP}^1$. By the universal property our homotopy extends by the identity on $\mathbb{RP}^1$ to finish the proof.

Question 3

Let C, D be chain complexes of Abelian groups such that $H_3(D)$ and $H_2(C)$ are trivial. Assume the following diagram commutes and p and r are surjective.

$$\begin{array}{ccccccc} C_4 & \longrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ D_4 & \longrightarrow & D_3 & \longrightarrow & D_2 & \longrightarrow & D_1 \end{array}$$

Prove that if the restriction of s to the image of C_2 is injective then q is surjective.

Take $d_3 \in D_3$ and consider $\partial d_3 = r(c_2)$ for some $c_2 \in C_2$ by surjectivity of r. $s(\partial c_2) = \partial r(c_2) = \partial\partial d_3 = 0$ so by injectivity of s restricted to the image of C_2 we have $\partial c_2 = 0$. Since $H_2(C) = 0$ we have some $c_3 \in C_3$ such that $\partial c_3 = c_2$. Now $\partial(q(c_3) - d_3) = r\partial(c_3) - r(c_2) = 0$ so since $H_3(D) = 0$ we must have some $d_4 \in D_4$ such that $\partial d_4 = q(c_3) - d_3$. By surjectivity of p find $c_4 \in C_4$ such that $p(c_4) = d_4$. Finally $q\partial c_4 = \partial p c_4 = \partial d_4 = q(c_3) - d_3$ so $d_3 = q(c_3 - \partial c_4)$.

Question 4

a. Explain how and for which values of m, n the cellular approximation theorem implies that $\pi_n(S^m)$ is trivial.

b. For a CW complex X prove that the inclusion of the n-skeleton into X induces a surjection $\pi_n(X, p) \to \pi_n(X, X^n, p)$. Here $p \in X^n$ is the base point.

Part a. We give all k-spheres the CW structure with a single 0-cell and a single k-cell. According to the cellular approximation theorem a map f representing an element of π_n is homotopy equivalent to a cellular map. When $n < m$ this means f is homotopy equivalent to a constant map since it must send the n-skeleton into the n-skeleton of S^m which is a point.
Part b. First the inclusion induces isomorphism on π_{n-1} because the cellular approximation theorem says any representant of an element of $\pi_{n-1}(X)$ is homotopy equivalent to a map into the n-skeleton. Finally surjectivity on π_n follows from the long exact sequence of homotopy groups for the pair (X, X^n), which says the following is exact: $\pi_n(X, p) \to \pi_n(X, X^n, p) \to \pi_{n-1}(X^n, p) \to \pi_{n-1}(X, p)$ Since the last map is an isomorphism the image of the second map is trivial which means its kernel and hence the image of the first map is the whole space.

Question 5

Compute all the homology groups of any CW complex X with the following properties. $X^0 = \{p, q\}$ and X^1 is obtained from X^0 by attaching two 1-cells, in both cases with surjective attaching map. Finally, for every positive integer n there are 3^n cells of dimension 3^n.

We can compute the homology groups using the homology of the cellular chain complex of X. It looks like $\cdots \to \mathbb{Z}^3 \to 0 \to \mathbb{Z}^2 \to \mathbb{Z}^2$ since the k-cells form a basis for the k-th chain group. All differentials are equal to 0 because either domain or range is 0, except the last one, from the space spanned by the two 1-cells to the space spanned by p, q. Since the attaching map is surjective for both 1-cells and the differential is given by taking the boundary, we see it has a kernel of rank 1 and so $H_1 = H_0 = \mathbb{Z}$. Given the other differentials were zero in other dimensions, the chain groups are isomorphic to the singular homology groups.