Solutions of the exercise hour - Week 4

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Exercise 1. *Let X be a topological space and A an abelian group. Let $U, V \subset X$ satisfying $X = \overset{\circ}{U} \cup \overset{\circ}{V}$, and let*

$$\begin{array}{ccccc} & & U & & \\ & \overset{i^U}{\nearrow} & & \overset{j^U}{\searrow} & \\ U \cap V & & & & X \\ & \underset{i^V}{\searrow} & & \underset{j^V}{\nearrow} & \\ & & V & & \end{array}$$

be the natural inclusion maps. Define groups homomorphisms

$$\partial_n : H_n(X; A) \to H_{n-1}(U \cap V; A)$$

such that the sequence of groups homomorphisms

$$\cdots \longrightarrow H_n(U \cap V; A) \xrightarrow{(i_*^U, i_*^V)} H_n(U; A) \oplus H_n(V; A) \xrightarrow{\begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}} H_n(X; A) \xrightarrow{\partial_n}$$
$$H_{n-1}(U \cap V; A) \longrightarrow H_{n-1}(U; A) \oplus H_{n-1}(V; A) \longrightarrow H_{n-1}(X; A) \xrightarrow{\partial_{n-1}}$$
$$H_{n-2}(U \cap V; A) \longrightarrow \cdots$$

is exact. This is called the ***Mayer-Vietoris sequence*** *of the cover $\{U, V\}$*

Proof. Consider the sequence of chain maps that levelwise is

$$0 \longrightarrow C_n(U \cap V; A) \xrightarrow{(i_*^U, i_*^V)} C_n(U; A) \oplus C_n(V; A) \xrightarrow{\begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}} C_n(\mathcal{S}_{\mathcal{O}}; A) \longrightarrow 0 \qquad (*)$$

and note that it is enough to check that it is exact, because since by hypothesis the cover $\mathcal{O} := \{U, V\}$ is admissible, the inclusion $\mathcal{S}_{\mathcal{O}}(X)$ in $\mathcal{S}(X)$ induces an isomorphism (small simplices theorem)

$$H_n(C(\mathcal{S}_{\mathcal{O}}(X); A)) \xrightarrow{\sim} H_n(X; A),$$

and therefore the long exact sequence of the previous sequence $(*)$ will give the requested diagram (and the morphisms ∂_n are precisely the composite of the connecting homomorphisms coming from the corresponding long exact sequence with the previous isomorphism coming from the small simplices theorem). Note that here we must also take into account that $H_n(C \oplus D; A) = H_n(C; A) \oplus H_n(D; A)$ for chain complexes C, D (exercise 5, sheet 1).

For the sake of simplicity, call $i = (i_*^U, i_*^V)$ and $j = \begin{pmatrix} j_*^U \\ -j_*^V \end{pmatrix}$ in $(*)$, and let us check its exactness:

- i injective:

  Note that i injective iff i_*^U injective and i_*^V injective, but both are because they precisely are the induced morphisms by the inclusions.

- j surjective:

  If $a\sigma \in C_n(\mathcal{S}_{\mathcal{O}}(X); A)$, then $\sigma \in \mathcal{S}(U)_n$ or $\sigma \in \mathcal{S}(V)_n$; so $a\sigma$ is the image of either $a\sigma$ or $-a\sigma$, respectively. For a finite sum of elements of $C_n(\mathcal{S}_{\mathcal{O}}(X); A)$ we just consider the corresponding sum of elements in the direct sum.

- $\operatorname{Im} i = \operatorname{Ker} j$:

  If $a\sigma \in C_n(U \cap V; A)$, i sends it to $(a\sigma, a\sigma) \in \operatorname{Im} i$, and by j it goes to 0, and linearity proves it for a sum. Conversely, if an element $(a\sigma, b\sigma')$ is in the kernel of j, then $a\sigma - b\sigma = 0 \in C_n(\mathcal{S}_{\mathcal{O}}(X); A)$. This implies that $\sigma = \sigma'$ and $a = b$; hence $\sigma \in \mathcal{S}(U \cap V)_n$.

□

Remark. The Mayer-Vietoris sequence also holds for de Rham cohomology: for a smooth manifold M of dimension n, the differential of forms d induces a *cochain complex*

$$0 \to \mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

(the "*co*" means that the degree goes up), because $d^2 = 0$, and one can take *co*homology groups $H^k_{dR}(M) := \operatorname{Ker} d / \operatorname{Im} d$. For two open sets $U, V \subset M$ such that $U \cup V = M$, the inclusions

$$U \cap V \rightrightarrows U \coprod V \to M$$

induce exact sequences

$$0 \to \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \to 0$$

which induce the Mayer-Vietoris sequence in de Rham cohomology.

Exercise 2. *Compute the homology groups $H_n(\mathbb{S}^m; A)$ using the Mayer-Vietoris sequence.*

Proof. This is a very easy exercise and I will not spell it out. The proof goes by induction: for $m = 0$ it boils down to the computation of the homology groups of the one-point space. For $m > 0$, one take the open sets

$$U = \{(x_0, \dots, x_m) \in \mathbb{S}^m : x_m > -1/3\}$$

and

$$V := \{(x_0, \dots, x_m) \in \mathbb{S}^m : x_m < 1/3\},$$

which are contractible; and one has that $U \cap V$ is homotopy equivalent to $\mathbb{S}^{m-1}$. □

Exercise 3. *Let X be a topological space, A an abelian group and $n \geq 0$ an integer. Let $X' \subseteq X$ be a non-empty closed subspace of X that is a neighbourhood deformation retract. As usual, we write X/X' for the quotient of X that identifies all points in X'.*

(i) Show that the quotient map $X \to X/X'$ induces an isomorphism of relative homology groups

$$H_n(X, X'; A) \xrightarrow{\sim} H_n(X/X', X'/X'; A).$$

(ii) Show that the relative homology groups $H_n(X, X'; A)$ are isomorphic to the homology groups $H_n(X/X'; A)$, if $n \geq 1$, and to $H_0(X/X'; A)/A$ if $n = 0$.

Proof. (i). Let us consider the following commutative diagram, where U is the neighbourhood of X' such that X' is a deformation retract of U (and therefore X' is homotopy equivalent to U):

$$\begin{array}{ccccc}
H_n(X, X'; A) & \xrightarrow{f} & H_n(X, U; A) & \xleftarrow{g} & H_n(X - X', U - X'; A) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow & & \downarrow{\scriptstyle \widetilde{\pi_*}} \\
H_n(X/X', X'/X'; A) & \xrightarrow{h} & H_n(X/X', U/X'; A) & \xleftarrow{i} & H_n(X/X' - X'/X', U/X' - X'/X'; A)
\end{array}$$

We will show that π_* is an isomorphism by showing that the rest of labelled arrows in the diagrams are too. By the "2 out of 3" property in the following diagram, we see that f is an isomorphism,

$$\begin{array}{ccccc} H_n(X';A) & \longrightarrow & H_n(X;A) & \longrightarrow & H_n(X,X';A)\\ \downarrow\wr & & \downarrow\wr & & \downarrow f\\ H_n(U;A) & \longrightarrow & H_n(X;A) & \longrightarrow & H_n(X,U;A)\end{array}$$

because the inclusion $X' \to U$ is a homotopy equivalence. Moreover, the homotopy $H : U \times I \to U$ defines another homotopy $\bar{H} : U/X' \times I \to U/X'$, continuous and well defined by the universal property of the quotient topology,

$$\begin{array}{ccccc} U\times I & \xrightarrow{H} & U & \longrightarrow & U/X' \\ \downarrow & & & \nearrow_{\bar{H}} & \\ U/X'\times I & & & & \end{array}$$

since for $x'_1, x'_2 \in X'$, we have $H(x'_1, t) \in X'$, $H(x'_2, t) \in X'$, and hence they are the same point in the quotient U/X'. In particular, the previous diagram says that X'/X' is a deformation retract of U/X' with homotopy $\bar{H}$. One concludes that h is isomorphism.

But we also have, by excision theorem, that g and i are also isomorphisms. For that just note that one can apply the theorem since $\overline{X'} = X' \subset O = \overset{\circ}{O} \subseteq \overset{\circ}{U}$, where O is the open set such that $X' \subseteq O \subseteq U$; and the same analogously with the quotient space (because the definition of the quotient topology).

Finally, it turns out that $\widetilde{\pi_*}$ is also isomorphism trivially, since

$$\widetilde{\pi}_{|X-X'} : X - X' \to (X - X')/X' = X/X' - X'/X',$$

$$\widetilde{\pi}_{|U-X'} : U - X' \to (U - X')/X' = U/X' - X'/X',$$

are homeomorphisms (there's nothing to identify!).

Altogether, one has that π_* is a group isomorphism because it is composite of isomorphisms.

(ii). Just note that the previous isomorphism tells us that

$$H_n(X, X'; A) = H_n(X/X', X'/X'; A) = H_n(X/X', *; A) = \begin{cases} H_n(X/X'; A) & n > 0 \\ H_0(X/X'; A)/H_0(*; A) & n = 0 \end{cases},$$

where in the last equality we have used the long exact sequence of the pair $(X/X', *)$. □