Practice exam - "Algebraic Topology"

Each question has a value of 2.5 points, and the value of each item is also indicated. You are allowed (and strongly encouraged!) to use concepts or statements of previous exercises in the next ones even though you didn't respond to them.

Good luck! - Sucess! - Buena suerte! - Buona fortuna! - Bonne chance! - Viel Glück! - Καλή τύχη!

1. Let C be a chain complex of abelian groups.

   (a) (0.5 pts) We say that C is **acyclic** if $H_n(C) = 0$ for all $n > 0$.
   Show that C is acyclic if and only if the sequence
   $$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$
   is exact.

   (b) (1 pt) Let C, C', C'' be chain complexes, and let $f, g : C \longrightarrow C'$ and $\bar{f}, \bar{g} : C' \longrightarrow C''$ be chain maps. Show that if f and g are chain homotopic, and $\bar{f}, \bar{g}$ are chain homotopic, then $\bar{f} \circ f$ is chain homotopic to $\bar{g} \circ g$.

   (c) (0.5 pts) We say that C is **contractible** if there is a chain homotopy P from the identity of C $\mathrm{Id} : C \longrightarrow C$ to the zero chain map $0 : C \longrightarrow C$.
   Show that every contractible chain complex is acyclic.

   (d) (0.5 pts) Show that the chain complex
   $$\cdots 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}/2\mathbb{Z},$$
   where $\partial_2(n) = 2n$ and $\partial_1(n) = [n]$, is acyclic but is not contractible. (*Hint:* Argue at degree 0).

2. Let (X, X') be a pair of spaces and let A be an abelian group.

   (a) (0.5 pts) Show that every element $[c] \in H_n(X, X'; A)$ is represented by an n-chain $c \in C_n(X; A)$ such that $\partial_n c \in C_{n-1}(X'; A)$, where we see $C_{n-1}(X'; A)$ as a subgroup of $C_{n-1}(X; A)$.

   (b) (1 pt) Show that $[c] = 0 \in H_n(X, X'; A)$ if and only if $c = \partial_{n+1} d + c'$ for chains $d \in C_{n+1}(X; A)$ and $c' \in C_n(X'; A)$.

   (c) (1 pt) Let $x \in X'$. Show that if $H_n(X', \{x\}; A) \cong 0$, then the map of pairs $(X, \{x\}) \longrightarrow (X, X')$ induces an isomorphism
   $$H_n(X, \{x\}; A) \cong H_n(X, X'; A).$$

3. Let X be the topological space obtained from two 2-spheres $\mathbb{S}^2$ by identifying their north poles to one point and their south poles to one point. That is, $X = (\mathbb{S}^2 \amalg \mathbb{S}^2)/\sim$, where n_i, s_i are the north and south pole of each sphere and the equivalence relation is $n_1 \sim n_2$ and $s_1 \sim s_2$.

   (a) (0.5 pts) Give a CW-structure on X.

   (b) (1 pt) Compute all integral homology groups $H_n(X; \mathbb{Z})$.

   (c) (1 pt) Let $p : \mathbb{S}^2 \amalg \mathbb{S}^2 \longrightarrow X$ and $q : \mathbb{S}^2 \longrightarrow \mathbb{RP}^2$ and be the projections to the quotient spaces and consider the map $q \vee q : \mathbb{S}^2 \amalg \mathbb{S}^2 \longrightarrow \mathbb{RP}^2 \vee \mathbb{RP}^2$. By the universal property of the quotient topology, there exists a unique continuous map $f : X \longrightarrow \mathbb{RP}^2 \vee \mathbb{RP}^2$ such that $f \circ p = q \vee q$.
   Compute the map $f_* : H_1(X; \mathbb{Z}) \longrightarrow H_1(\mathbb{RP}^2 \vee \mathbb{RP}^2)$.

4. Let (X, X') be a pair of spaces.

   (a) (0.5 pts) Show that, if (X, X') has the HEP, so does $(X \times Y, X' \times Y)$ for any topological space Y.

   (b) (1 pt) Let X be a CW-complex and let $x_0 \in X$ be a 0-cell. Let SX be the (unreduced) **suspension** of X, that is, the quotient $X \times [0, 1]$ by identifying the subspace $X \times \{0\}$ to one point and $X \times \{1\}$ to one point; and let ΣX be the **reduced suspension** of X, that is, the quotient of $X \times [0, 1]$ by identifying the subspace $X \times \{0\}$ to one point, the subspace $X \times \{1\}$ to one point and the subspace $\{x_0\} \times I$ to one point.

   Show that SX and ΣX are homotopy equivalent.

   (c) (1 pt) Let X be a path-connected finite 1-dimensional CW-complex with c_0 0-cells and c_1 1-cells.

   Show that X is homotopy equivalent to a wedge sum of $c_1 - c_0 + 1$ circles $\mathbb{S}^1$. (*Hint:* Use induction over the number of 0-cells).